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SPECTRAL LEGENDRE AND CHEBYSHEV APPROXIMATION FOR THE STOKES INTERFACE PROBLEMS

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ABSTRACT. The numerical solution of the Stokes equation with discontinuous viscosity and singular force term is challenging, due to the discontinuity of pressure, non-smoothness of velocity, and coupled discontinuities along interface.In this paper, we give an efficient algorithm to solve this problem by employing spectral Legendre and Chebyshev approximations.First, we present the algorithm for a problem defined in rectangular domain with straight line interface. Then it is generalized to a domain with smooth curve boundary and interface by employing spectral element method. Numerical experiments demonstrate the accuracy and efficiency of our algorithm and its spectral convergence.

1. INTRODUCTION

Stokes and Navier-Stoles equations with discontinuous viscosity and singular forces have several applications in science and engineering. The flow pattern of blood in the heart [14] is a typical one of many examples. In this paper we present a simple and easy to implement, but efficient algorithm to solve this problem numerically. To state the problem, let Ω be an open bounded domain in $\mathbb{R}^2$ and Γ be a curve separating the domain Ω into two sub-domains Ω^+ and Ω^-, such that $\overline{\Omega} = \overline{\Omega^+} \cup \overline{\Omega^-} \cup \Gamma$. We refer to Γ as *interface*. The boundary of Ω is denoted by $\partial\Omega$ and also $\partial\Omega^\pm = \overline{\Omega^\pm} \cap \partial\Omega$. We consider the Stokes equation with discontinuous viscosity across the interface and singular force along the interface, that can be written as

$$\begin{cases} -\nu\Delta \mathbf{u} + \nabla p = \mathbf{f} + \mathbf{g}\delta_\Gamma, & \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} = 0, & \text{in} \quad \Omega, \\ \mathbf{u} = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

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where $\mathbf{u}$ is the velocity vector, p is the pressure, $\mathbf{f}$ is an external force function, $\mathbf{g}$ is a force density defined only on the interface Γ and δ_Γ is the 2-dimensional delta function with support along the interface Γ. We assume that the viscosity ν is a piecewise constant defined by

$$\nu(x,y) = \begin{cases} \nu^+, & (x,y) \in \Omega^+ \\ \nu^-, & (x,y) \in \Omega^-. \end{cases}$$

The uniqueness of p can be achieved by imposing average zero, i.e.,

$$\int_\Omega p dx = 0.$$

We call this as *"Stokes interface problem"*. The existence and uniqueness of the weak solution of (1.1) can be found in [17]. It is well known that pressure is discontinuous and velocity continuous but non-smooth along the interface, due to the presence of singular source term and discontinuous viscosity. Several methods have been proposed for the case of continuous viscosity with singular source term (See [12] and references therein). However, for discontinuous viscosity, the jump condition for velocity and pressure is coupled, and approximating the solution is problematic. To get accurate numerical approximation, optimal interface conditions ([8, 12]) are necessary. Peskin's immersed boundary model that was introduced to simulate the blood flow in a human's heart [14] is one of the most successful Cartesian grid methods. LeVeque and Li [12] proposed immersed interface method for Stokes flows which has the second order accuracy. The authors in [9] introduced two augmented variables that are defined only along the interface so that the jump conditions can be decoupled and immersed interface method can be applied [11]. They get second order immersed interface method using finite difference discretization. Rutka [18] developed the explicit immersed interface method (EJIIM) for two-dimensional Stokes flows on irregular domains which is up to second order derivatives along the interface. The authors in [19] using finite volume method, reshaped immersed boundary cells and used polynomial interpolating functions to approximate the fluxes and gradients on the faces of the boundary cells which is second order accurate.

However, the interface conditions which have been used in above works, include coupled interface condition for pressure and velocity, as well as zeros, first and second order derivatives of velocity and pressure. The numerous number of interface conditions and being coupled cause an expensive computational. The presented algorithm here is generalization of our previous work for elliptic interface problems [5]. The advantages of this method beside its spectral accuracy, is that we use only two interface conditions, one for continuity of velocity and the other is coupled interface condition for pressure and velocity in which only the first derivative of velocity is involved. Pseudo-spectral method also have been used to approximate solution of 1-dimensional elliptic interface problem [16] and 2-dimensional one [5, 15].

The content of the paper is organized as follows. Interface condition are derived in section 2. The pseudo-spectral algorithm is presented in section 3. Numerical examples are given in section 4 to show efficiency of the proposed algorithm. The paper is finalized with concluding remarks in section 5.

2. INTERFACE CONDITIONS

This section concerns deriving interface conditions. As stated earlier, interface condition has an significant role for obtaining an accurate numerical approximation. In order to derive interface conditions for the Stokes interface problem, we use finite element principle to the Stokes equation (1.1). The weak formulation of the Stokes equation reads:

$$\text{seek} \quad (\mathbf{u},\, p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega) \quad \text{such that}$$

$$\nu \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} - \int_\Omega p\,(\nabla \cdot \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} + \int_\Gamma \mathbf{g}\mathbf{v} ds, \qquad \forall \mathbf{v} \in H_0^1(\Omega)^2, \tag{2.1}$$

$$-\int_\Omega q\,(\nabla \cdot \mathbf{u}) = 0, \qquad \forall q \in L_0^2(\Omega). \tag{2.2}$$

It is well known that pressure is discontinuous and velocity is continuous with discontinuous derivatives along the interface. Suppose that $\mathbf{u}$ and p are smooth in each sub-domain $\Omega^\pm$. We apply Green's formula to (2.1) to get

$$-\int_{\Omega^+} [\nabla \cdot (\nu \nabla \mathbf{u})] \cdot \mathbf{v} - \int_{\Omega^-} [\nabla \cdot (\nu \nabla \mathbf{u})] \cdot \mathbf{v} + \int_{\Omega^+} \nabla p \cdot \mathbf{v} + \int_{\Omega^-} \nabla p \cdot \mathbf{v}$$
$$- \int_\Gamma \big([p \cdot \mathbf{n} - \nu \nabla \mathbf{u} \cdot \mathbf{n}] + \mathbf{g}\big) \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v}.$$

Here, the interface jump is defined as follows

$$[v]_\Gamma = v^+ - v^-,$$

where v^+ and v^- are the traces of $v|_{\Omega^+}$ and $v|_{\Omega^-}$, respectively, on Γ. Then, from the above equation, we have the following strong equations:

$$\begin{cases} -\nu^\pm \Delta \mathbf{u}^\pm + \nabla p^\pm = \mathbf{f}^\pm, & \text{in} \quad \Omega^\pm, \\ \nabla \cdot \mathbf{u}^\pm = 0, & \text{in} \quad \Omega^\pm, \\ \mathbf{u}^\pm = 0, & \text{on} \quad \partial\Omega^\pm, \end{cases} \tag{2.3}$$

along with the following jump condition:

$$[\mathbf{u}]_\Gamma = 0, \qquad [\nu \nabla \mathbf{u} \cdot \mathbf{n} - p \cdot \mathbf{n}]_\Gamma = \mathbf{g}, \tag{2.4}$$

where $\mathbf{n} = (n_1, n_2)$ denotes the unit normal vector on interface pointing into Ω^-. It should be noted that in our numerical algorithm given in section 3, we use only two interface conditions (2.4), to get spectral accuracy.

3. PSEUDO-SPECTRAL METHOD

In this section we present an algorithm for solving Stokes interface problem by pseudo-spectral method [5, 6, 10]. First, we start with a problem defined on the rectangle domain with straight line interface and then we extend the algorithm to problems with arbitrary domain in which the boundary and interface curves are smooth. We give some simple facts about pseudo-spectral method. We use standard notations and definitions for the weighted Sobolev spaces $H_\omega^s(\Omega)$ equipped with weighted inner product $(\cdot, \cdot)_{s,\omega}$ and corresponding weighted norms $\| \cdot$

$\|_{s,\omega}$, $s \geq 0$, where $\omega(x,y) = \hat{\omega}(x)\hat{\omega}(y)$ is the Legendre weight function when $\hat{\omega}(t) = 1$ and Chebyshev weight function when $\hat{\omega}(t) = 1/\sqrt{1-t^2}$. Let $\mathcal{P}_N$ be the space of all polynomials of degree less than or equal to N and let $\{\xi_i\}_{i=0}^N$ be the Legendre Gauss Lobatto (LGL) or Chebyshev Gauss Lobatto (CGL) points on $[-1,1]$ such that $-1 =: \xi_0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N := 1$. For Legendre case, $\{\xi_i\}_{i=0}^N$ are zeros of $(1-t^2)L_N'(t)$ where L_N is the N^{th} Legendre polynomial and the corresponding quadrature weights $\{w_i\}_{i=0}^N$ are given by

$$w_0 = w_N = \frac{2}{N(N+1)}, \quad w_j = \frac{2}{N(N+1)} \frac{1}{[L_N(\xi_j)]^2}, \quad 1 \leq j \leq N-1. \tag{3.1}$$

For Chebyshev case, $\{\xi_i\}_{i=0}^N$ are zeros of $(1-t^2)T_N'(t)$ where T_N is the N^{th} Chebyshev polynomial and the corresponding quadrature weights $\{w_i\}_{i=0}^N$ are given by

$$w_0 = w_N = \frac{\pi}{2N}, \quad w_j = \frac{\pi}{N}, \quad 1 \leq j \leq N-1. \tag{3.2}$$

For any continuous function u on $[-1,\ 1]$, let $I_N u$ denote its Lagrange interpolation at collocation points $\{\xi_i\}_{i=0}^N$, i.e.,

$$I_N u(\xi_i) = u(\xi_i), \qquad i = 0, 1 \cdots, N.$$

Let $\{\psi_j\}_{j=0}^N \subset \mathcal{P}_N$ be the Lagrange basis functions of degree N such that

$$\psi_j(\xi_k) = \delta_{jk} \quad \forall\, j,k = 0,1,\cdots,N.$$

Then

$$I_N u(x) = \sum_{j=0}^N u(\xi_j)\psi_j(x).$$

The pseudo-spectral derivative $\partial_N u$ of a continuous function u is defined to be the exact derivative of the interpolant of u, that is

$$\partial_N u(\xi_i) = \sum_{j=0}^N u(\xi_j)\psi_j'(\xi_i).$$

Then the pseudo-spectral derivative matrix D_N is

$$D_N(i,j) := \psi_j'(\xi_i).$$

Let U be the vector valued function containing nodal values of u at ξ_j, i.e. $U = (u(\xi_0), \cdots, u(\xi_N))^T$, then the derivative vector valued function of u' is $D_N U = (u'(\xi_0), \ldots, u'(\xi_N))^T$. If the interval $[-1\,,\ 1]$ is replaced by $[a\,,\ b]$, then we can use the following linear transformation

$$t = \frac{b-a}{2}(x+1) + a\ : [-1,1] \to [a,b]$$

to find Gauss-points $\{\hat{\xi}_j\}_{j=0}^N$ and the quadrature weights $\{\hat{w}_j\}_{j=1}^N$, respectively given by

$$\hat{\xi}_j = \frac{b-a}{2}(\xi_j + 1) + a \quad \text{and} \quad \hat{w}_j = \frac{b-a}{2} w_j.$$

Introducing

$$\hat{\psi}_j(t) := \psi_j(x) = \psi_j\left(\frac{2}{b-a}(t-a) - 1\right),$$

we obtain the following spectral matrix $\hat{D}_N$

$$\hat{D}_N = \frac{2}{b-a}\, D_N \quad \text{where} \quad \hat{D}_N(i,j) := \hat{\psi}_j'(\hat{\xi}_i) = \frac{2}{b-a}\, \psi_j'(\xi_i).$$

The two-dimensional LGL and CGL nodes $\{\mathbf{x}_{ij}\}$ and weights $\{\mathrm{w}_{ij}\}$ are defined as

$$\mathbf{x}_{ij} = (\xi_i, \xi_j), \quad \mathrm{w}_{ij} = w_i w_j, \quad i, j = 0, 1, \cdots, N.$$

Let $\mathcal{Q}_N$ be the space of polynomials of degree less than or equal to N with respect to each variable x and y. The basis functions are also defined

$$\psi_{ij}(x,y) = \psi_i(x)\, \psi_j(y), \qquad i,\, j = 0, 1, \ldots, N.$$

We reorder the LGL and CGL points from bottom to top and then from left to right such that $\mathbf{x}_{k(N+1)+l} := \mathbf{x}_{kl} = (\xi_k,\, \xi_l)$ for $k, l = 0, 1, \cdots, N$. Then pseudo-spectral derivative matrix in 2-dimensional space is defined via the Kronecker tensor product, that is

$$S_x = D_N \otimes I_N, \quad S_y = I_N \otimes D_N,$$
$$S_{xx} = D_N^2 \otimes I_N, \quad S_{yy} = I_N \otimes D_N^2,$$

where I_N denotes the identity matrix of the same order as D_N.

Here we consider pseudo-spectral method for problem (2.3) and (2.4) with discontinuous viscosity ν and singular force. Let $\Omega = (a,b) \times (c,d)$ be a quadrilateral domain and $\Gamma = \{\alpha\} \times (c,d)$ be an interface separating the domain Ω into two sub-domains Ω^+ and Ω^-, as depicted in FIGURE 1.

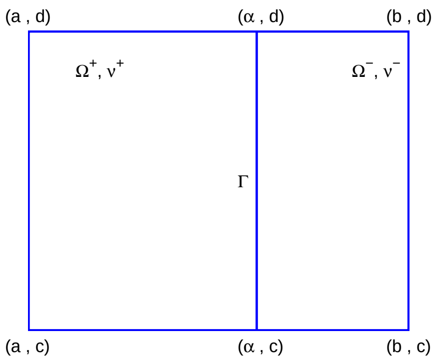

FIGURE 1. Schematic of domain Ω, its subdomains $\Omega^{\pm}$ and interface Γ for Stokes interface problem with discontinuous viscosity ν and singular force along interface.

Suppose that $v \in \mathcal{Q}_N$ is the pseudo-spectral approximation solution to problem (2.3)-(2.4) where v could be velocity or pressure approximation. Then the approximation solution of the interface problem (2.3) and (2.4) can be expressed by

$$v^{\pm}(x,y) = \sum_{i=0}^{N}\sum_{j=0}^{N} v_{ij}^{\pm}\ \psi_{ij}^{\pm}(x,y).$$

Although it is possible to use different polynomial order approximation on each subdomain, we use the same polynomial order approximation, for the sake of simplicity. Suppose that $\mathbf{u}^{\pm} \in \mathcal{Q}_N^2 \cap H_0^1(\Omega^{\pm})^2$ and $p^{\pm} \in \mathcal{Q}_M \cap L_0^2(\Omega^{\pm})$ are respective pseudo-spectral approximation of velocity and pressure of the Stokes interface problem (2.3) and (2.4). In this paper, we take $M = N-2$ for the compatibility (or inf-sup) condition([1, 2, 13]).

Hence we have the following equations

$$\begin{cases} -\nu^{\pm}\Delta\mathbf{u}^{\pm}(\xi_i,\xi_j) + \nabla p^{\pm}(\xi_i,\xi_j) = \mathbf{f}(\xi_i,\xi_j), & \forall(\xi_i,\xi_j) \in \Omega^{\pm},\, i,j = 0,1,\dots,N, \\ \nabla\cdot\mathbf{u}^{\pm}(\xi_k,\xi_l) = 0, & \forall(\xi_k,\xi_l) \in \Omega^{\pm},\, k,l = 0,1,\dots,M, \end{cases} \tag{3.3}$$

which can be written in matrix-vector form as

$$\mathbf{A}_1\mathbf{X} = \left[\begin{array}{c|c} \mathbf{S}^+ & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{S}^- \end{array}\right] \left[\begin{array}{c} \mathbf{X}^+ \\ \hline \mathbf{X}^- \end{array}\right] = \left[\begin{array}{c} F^+ \\ \hline F^- \end{array}\right] = F_1,$$

where,

$$\mathbf{S}^{\pm} = \begin{bmatrix} -\nu^{\pm}(S_{xx}^{\pm} + S_{yy}^{\pm}) & \mathbf{0} & \hat{S}_x^{\pm} \\ \mathbf{0} & -\nu^{\pm}(S_{xx}^{\pm} + S_{yy}^{\pm}) & \hat{S}_y^{\pm} \\ \hat{S}_x^{\pm} & \hat{S}_y^{\pm} & \mathbf{0} \end{bmatrix},\ F^{\pm} = \begin{bmatrix} f_1^{\pm}(\xi_i,\xi_j) \\ f_2^{\pm}(\xi_i,\xi_j) \\ \mathbf{0} \end{bmatrix}$$

and $\mathbf{X}^{\pm} = \left[U_1^{\pm},\ U_2^{\pm},\ P^{\pm}\right]^t$. Here

$$S_{xx}^{+} = \left(\frac{2}{\alpha - a}\right)^2 D_N^2 \otimes I_N, \quad S_{yy}^{+} = \left(\frac{2}{d-c}\right)^2 I_N \otimes D_N^2,$$
$$S_{xx}^{-} = \left(\frac{2}{b - \alpha}\right)^2 D_N^2 \otimes I_N, \quad S_{yy}^{-} = \left(\frac{2}{d-c}\right)^2 I_N \otimes D_N^2,$$
$$\hat{S}_x^{+} = \left(\frac{2}{\alpha - a}\right) \tilde{S}_x^{+}, \quad \hat{S}_y^{+} = \left(\frac{2}{d-c}\right) \tilde{S}_y^{+},$$
$$\hat{S}_x^{-} = \left(\frac{2}{b - \alpha}\right) \tilde{S}_x^{-}, \quad \hat{S}_y^{-} = \left(\frac{2}{d-c}\right) \tilde{S}_y^{-},$$

where $\tilde{S}_t^{\pm}$, $(t = x$ or $y)$ is pseudo-spectral derivative matrix of velocity interpolated at nodal points of pressure, and $U_1,\ U_2$ and P are vectors containing the nodal values of functions $u_1,\ u_2$ and p, respectively.

To impose the jump and boundary conditions, let $v_{if}^{\pm}$, $v_{bd}^{\pm}$ and $v_{in}^{\pm}$ denote the approximation values of $v^{\pm}$ at nodal points on the interface, boundaries, and interior of domain $\Omega^{\pm}$, respectively. By jump conditions $[\mathbf{u}]_\Gamma = 0$ and $[\nu\nabla\mathbf{u}\cdot\mathbf{n} - p\cdot\mathbf{n}]_\Gamma = \mathbf{g}$, we have $\mathbf{u}_{if}^{+} - \mathbf{u}_{if}^{-} = 0$ and

$\nu^+\nabla\mathbf{u}_{if}^+\cdot\mathbf{n}-p_{if}^+\cdot\mathbf{n}-\nu^-\nabla\mathbf{u}_{if}^-\cdot\mathbf{n}+p_{if}^-\cdot\mathbf{n}=G$ for $\mathbf{x}_i$ on Γ respectively, where $G=(\,\mathbf{g}(\mathbf{x}_i)\,)$. The boundary conditions can be imposed as

$$\mathbf{u}_{bd}^{\pm}=0.$$

Now, the boundary and jump conditions can be represented in matrix-vector form as

$$\mathbf{A}_2\mathbf{X}=\left[\begin{array}{c|c} B_1 & \mathbf{0} \\ \hline B_2 & -B_2 \\ \hline B_3^+ & -B_3^- \end{array}\right]\left[\begin{array}{c}\mathbf{X}^+\\ \hline \mathbf{X}^-\end{array}\right]=F_2$$

where

$$B_1=\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0\end{bmatrix},\quad B_2=\begin{bmatrix} 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0\end{bmatrix},$$

$$B_3^{\pm}=\begin{bmatrix} 0 & 0 & \nu^{\pm}(n_1S_x^{\pm}+n_2S_y^{\pm}) & 0 & 0 & 0 & 0 & 0 & -n_1I\\ 0 & 0 & 0 & 0 & 0 & \nu^{\pm}(n_1S_x^{\pm}+n_2S_y^{\pm}) & 0 & 0 & -n_2I\end{bmatrix},$$

$$\mathbf{X}^{\pm}=\left[\mathbf{u}1_{bd}^{\pm},\,\mathbf{u}1_{in}^{\pm},\,\mathbf{u}1_{if}^{\pm},\,\mathbf{u}2_{bd}^{\pm},\,\mathbf{u}2_{in}^{\pm},\,\mathbf{u}2_{if}^{\pm},\,p_{bd}^{\pm},\,p_{in}^{\pm},\,p_{if}^{\pm}\right]^T,$$

and

$$F_2=[0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,0,\,0,\,G_1,\,0,\,0,\,G_2,\,0,\,0,\,0]^T.$$

Combining two systems, we have the following linear system

$$\mathbf{AX}=F$$

where $\mathbf{A}=\mathbf{A}_1+\mathbf{A}_2$ and $F=F_1+F_2$.

In this paper we use the following stabilization technique (See [1, 2, 13]). If the divergence operator is applied to the momentum equation in (2.3), owing to the fact that $\mathbf{u}$ is divergence free, we have

$$\Delta p^{\pm}=\nabla\cdot\mathbf{f}^{\pm}\quad\text{in }\Omega^{\pm}. \tag{3.4}$$

Using the equation (3.4), we stabilized the continuity equation as

$$\nabla\cdot\mathbf{u}^{\pm}-\gamma\Delta p^{\pm}=-\gamma\nabla\cdot\mathbf{f}^{\pm}\quad\text{in }\Omega$$

where γ is a constant. After stabilization, we have

$$\mathbf{S}^{\pm}=\begin{bmatrix} -\nu^{\pm}(S_{xx}^{\pm}+S_{yy}^{\pm}) & \mathbf{0} & \hat{S}_x^{\pm}\\ \mathbf{0} & -\nu^{\pm}(S_{xx}^{\pm}+S_{yy}^{\pm}) & \hat{S}_y^{\pm}\\ \hat{S}_x^{\pm} & \hat{S}_y^{\pm} & -\gamma(\tilde{S}_{xx}^{\pm}+\tilde{S}_{yy}^{\pm})\end{bmatrix},$$

$$F^{\pm}=\begin{bmatrix} f_1^{\pm}(\xi_i,\xi_j)\\ f_2^{\pm}(\xi_i,\xi_j)\\ -\gamma\tilde{S}_x^{\pm}f_1^{\pm}(\xi_k,\xi_l)-\gamma\tilde{S}_y^{\pm}f_2^{\pm}(\xi_k,\xi_l)\end{bmatrix}$$

where

$$\tilde{S}_x^{+}=\left(\frac{2}{\alpha-a}\right)D_M\otimes I_M,\quad \tilde{S}_y^{+}=\left(\frac{2}{d-c}\right)I_M\otimes D_M,$$

$$\tilde{S}_{xx}^{+} = \left(\frac{2}{\alpha - a}\right)^2 D_M^2 \otimes I_M, \quad \tilde{S}_{yy}^{+} = \left(\frac{2}{d - c}\right)^2 I_M \otimes D_M^2,$$
$$\tilde{S}_{xx}^{-} = \left(\frac{2}{b - \alpha}\right)^2 D_M^2 \otimes I_M, \quad \tilde{S}_{yy}^{-} = \left(\frac{2}{d - c}\right)^2 I_M \otimes D_M^2.$$

Since the matrix $-\gamma(\tilde{S}_{xx}^{\pm} + \tilde{S}_{yy}^{\pm})$ is symmetric positive definite, the matrix $\mathbf{S}^{\pm}$ become symmetric and positive definite. Hence the resulting algebraic system $\mathbf{AX} = F$ can be efficiently solved by direct and iterative methods.

Remark 1. *In the case of the Stokes interface problem defined on curved domain with curve interface, a mapping so-called Gordon-Hall transformation* [3, 4] *can be used to transform the domain and equations into a rectangular domain. For complete explanation and examples of Gordon-Hall transformation see* [5, 7]. *It should be noted that a great advantage of using Gordon-Hall map and pseudo-spectral method is that the collocation points always lie on the interface and two neighboring domains share the same nodes on the interface, regardless of interface shape.*

4. NUMERICAL RESULTS

In this section, we first give an example defined on rectangle domain with straight line interface as in FIGURE 1. And then we present some examples defined on more complicated domain which make us use pseudo-spectral element method to solve them. Denote by v_N the discrete solution of problem and by $e = v - v_N$, the errors for $v \in \{u_1,\ u_2,\ p\}$. We present their $L^2_w(\Omega)$ and $H^1_w(\Omega)$ discrete norm of error which is defined, respectively, as follows:

$$\|e\|_{w,N}^2 = \sum_{i,j=0}^{N} \mathrm{w}_{ij} e^2(\mathbf{x}_{ij}), \quad \|e\|_{1,w,N}^2 = \|\nabla e\|_{w,N}^2 + \|e\|_{w,N}^2.$$

Example 1 (Straight line interface)**.** *Consider the Stokes interface problem with the following exact solution*

$$u_1(x,y) = \begin{cases} x^2y^3 + \exp(y) + \sin(\pi y), & (x,y) \in \Omega^+, \\ \frac{2}{3}x^3 y + \exp(y) + \sin(\pi y), & (x,y) \in \Omega^-, \end{cases}$$

$$u_2(x,y) = \begin{cases} -\frac{1}{2}xy^4 + \cos(\pi x), & (x,y) \in \Omega^+, \\ -x^2y^2 + \cos(\pi x), & (x,y) \in \Omega^-, \end{cases}$$

$$p(x,y) = \begin{cases} x^2(y-1), & (x,y) \in \Omega^+, \\ (y-1)^3, & (x,y) \in \Omega^-, \end{cases}$$

where $\Omega^+ = [-1,0] \times [0,2],\ \Omega^- = [0,1] \times [0,2]$ *and* $\Gamma = \{0\} \times [0,2]$. *The singular source term* $\mathbf{g} = [\nu \nabla \mathbf{u} \cdot \mathbf{n} - p \cdot \mathbf{n}]_\Gamma$ *is given by using the above solutions.*

The errors in L^2_w and H^1_w-norm discretization for Legendre and Chebyshev approximation are given in Tables 1 and 2, respectively, which show the exponential rate of convergence with respect to N regardless of discontinuous viscosity and singular source term.

TABLE 1. Error discretization of Example 1 with $\nu^+ = 1$ and $\nu^- = 5$ for Legendre case.

	u_1		u_2		p	
N	$\|e\|_{L^2}$	$\|e\|_{H^1}$	$\|e\|_{L^2}$	$\|e\|_{H^1}$	$\|e\|_{L^2}$	$\|e\|_{H^1}$
6	9.3152e-04	7.9018e-03	1.5884e-03	1.1424e-02	1.6454e-01	3.4110e-01
10	8.3228e-08	4.8696e-07	1.4383e-07	9.0609e-07	9.3467e-06	4.5888e-05
14	1.5397e-12	9.5044e-12	2.6967e-12	1.7453e-11	9.1089e-10	2.5590e-09
18	2.6239e-14	3.9698e-13	1.9463e-14	6.6407e-13	1.0726e-09	1.2253e-09

TABLE 2. Error discretization of Example 1 with $\nu^+ = 1$ and $\nu^- = 5$ for Chebyshev case.

	u_1		u_2		p	
N	$\|e\|_{L^2}$	$\|e\|_{H^1}$	$\|e\|_{L^2}$	$\|e\|_{H^1}$	$\|e\|_{L^2}$	$\|e\|_{H^1}$
6	1.0644e-03	1.6923e-02	1.5043e-03	1.6272e-02	3.0275e-01	7.0536e-01
10	4.6437e-08	1.5634e-06	9.6710e-08	1.0340e-06	3.6578e-05	2.0598e-04
14	6.5587e-13	3.1726e-11	1.5445e-12	2.0782e-11	2.7876e-09	1.1828e-08
18	1.8977e-13	2.0595e-11	2.6345e-13	4.2578e-11	1.5852e-08	5.0606e-08

The exact solution, approximate solution and the error of pressure for Legendre case of this example are plotted in FIGURE 2.

Example 2. *Let the domain Ω be $\Omega = [-1, 1]\times[-1, 1]$ and the interface curve be $x^2+y^2 = \frac{1}{4}$. The decomposition of domain Ω is given in* FIGURE *3. The exact solutions are*

$$u_1(x,y) = \begin{cases} \dfrac{y}{r} - \dfrac{y}{r_0}, & \textit{if} \quad r>r_0, \\ 0, & \textit{if} \quad r \le r_0, \end{cases}$$

$$u_2(x,y) = \begin{cases} -\dfrac{x}{r} + \dfrac{x}{r_0}, & \textit{if} \quad r>r_0, \\ 0, & \textit{if} \quad r \le r_0, \end{cases}$$

$$p(x,y) = \begin{cases} \cos(\pi x)\cos(\pi y), & \textit{if} \quad r>r_0, \\ 0, & \textit{if} \quad r \le r_0, \end{cases}$$

where $r_0 = \frac{1}{2}$, and $r = \sqrt{(x^2+y^2)}$.

We note that along dashed line common sides, the conditions $[\mathbf{u}] = 0$, $[\nu\nabla\mathbf{u}\cdot\mathbf{n} - p\cdot\mathbf{n}] = 0$ hold. The errors in L^2_w and H^1_w-norm discretization for Legendre and Chebyshev

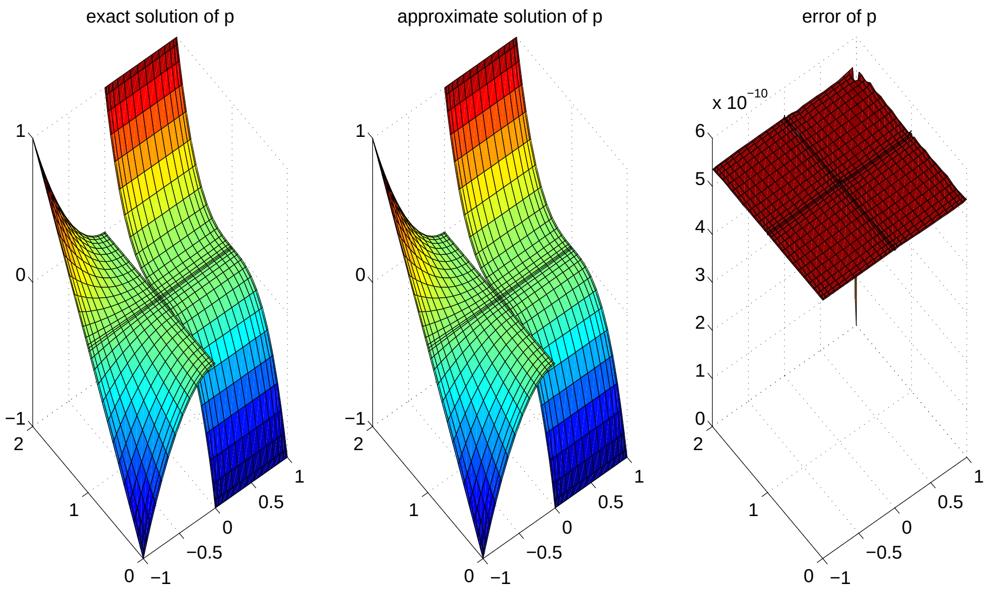

FIGURE 2. Exact and approximate solutions, and its error of pressure for $N = 18$ of Example 1.

approximation are given in Tables 3 and 4, respectively, which show the spectral convergence of the proposed algorithm. The exact solution, approximate solution and the error of pressure for Legendre case of this example are plotted in FIGURE 4.

TABLE 3. Error discretization of Example 2 for Legenre case, with $\nu^+ = 0.1$.

	u_1		u_2		p	
N	$\|e\|_{L^2}$	$\|e\|_{H^1}$	$\|e\|_{L^2}$	$\|e\|_{H^1}$	$\|e\|_{L^2}$	$\|e\|_{H^1}$
6	1.1195e-01	3.4194e+00	1.0728e-01	2.2860e+00	6.1792e-01	1.2231e+01
10	1.9327e-05	1.0323e-04	1.9193e-05	1.4186e-04	2.3266e-05	6.5461e-04
14	9.3725e-07	5.1590e-06	8.9838e-07	5.5775e-06	1.8696e-06	1.4162e-04
18	1.4382e-08	1.2166e-07	1.8357e-08	1.2493e-07	5.5682e-08	5.5863e-06
22	3.5617e-10	2.3544e-09	4.0336e-10	2.7923e-09	2.4732e-09	3.3634e-08
26	1.3608e-11	6.9920e-11	1.3506e-11	8.2877e-11	5.2944e-10	1.4156e-09
30	2.9087e-13	1.9949e-12	3.1573e-13	2.1529e-12	1.5060e-10	1.5746e-10

TABLE 4. Error discretization of Example 2 for Chebychev case, with $\nu^+ = 0.1$.

	u_1		u_2		p	
N	$\|e\|_{L^2}$	$\|e\|_{H^1}$	$\|e\|_{L^2}$	$\|e\|_{H^1}$	$\|e\|_{L^2}$	$\|e\|_{H^1}$
6	5.7439e-02	2.0566e+00	6.4792e-02	1.5999e+00	5.0340e-01	8.0266e+00
10	1.4296e-05	2.7028e-04	1.3009e-05	3.5747e-04	1.2682e-03	5.1796e-03
14	2.6209e-07	2.6194e-06	2.4317e-07	4.4447e-06	1.6483e-05	1.7272e-04
18	7.2807e-09	5.3576e-08	6.1460e-09	1.0181e-07	5.0464e-07	2.9558e-06
22	1.6049e-10	1.4202e-09	1.4739e-10	2.7203e-09	1.9297e-08	1.2404e-07
26	4.5497e-12	4.1539e-11	4.0185e-12	8.5765e-11	9.3411e-10	6.2526e-09
30	1.5859e-13	1.6333e-12	1.4411e-13	2.8550e-12	4.2261e-11	2.9465e-10

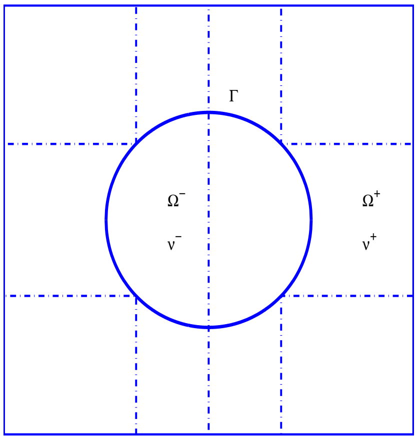

FIGURE 3. Schematic of domain Ω, interface Γ and its decomposition for Stokes interface problems in Examples 2 to 4.

Example 3. *We consider an example with smooth velocity and discontinuous pressure across interface where* $\Omega = [-2,\ 2] \times [-2,\ 2]$ *and the interface curve is the unit circle, i.e.,* $x^2 + y^2 = 1$. *The decomposition of domain* Ω *is given in* FIGURE *3. The exact solutions are*

$$\begin{cases} u_1(x,y) = y(x^2+y^2-1), & (x,y) \in \Omega, \\ u_2(x,y) = -x(x^2+y^2-1), & (x,y) \in \Omega, \end{cases}$$

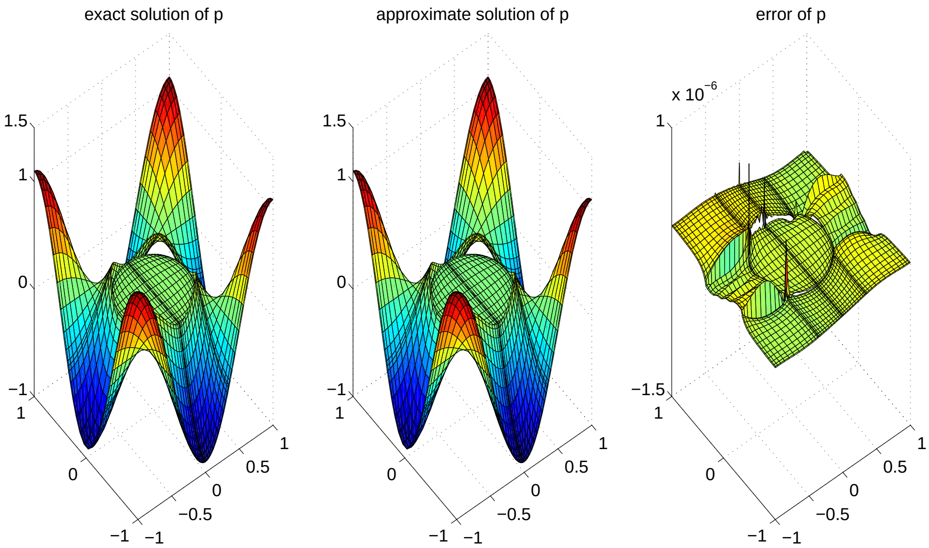

FIGURE 4. Exact and approximate solutions, and its error of pressure for $N = 18$ of Example 2.

$$p(x,y) = \begin{cases} 1, & (x,y) \in \Omega^-, \\ 0, & (x,y) \in \Omega^+. \end{cases}$$

The errors in L^2_w and H^1_w-norm discretization for Legendre and Chebyshev are given in Tables 5 and 6, respectively, which are evidence of exponential convergence of the given algorithm.

Example 4. *We consider the previous example with the same exact solution inside the unit circle, but the solutions are set to be zero outside the unit circle. That is*

$$u_1(x,y) = \begin{cases} y(x^2+y^2-1), & (x,y) \in \Omega^-, \\ 0, & (x,y) \in \Omega^+, \end{cases}$$

$$u_2(x,y) = \begin{cases} -x(x^2+y^2-1), & (x,y) \in \Omega^-, \\ 0, & (x,y) \in \Omega^+, \end{cases}$$

$$p(x,y) = \begin{cases} 1, & (x,y) \in \Omega^-, \\ 0, & (x,y) \in \Omega^+. \end{cases}$$

The errors in L^2_w and H^1_w-norm discretization for Legendre and Chebyshev are given in Tables 7 and 8, respectively, which spectral accuracy of the proposed method is evident. Exact

TABLE 5. Error discretization of Example 3 for Legenre case, with $\nu^+ = 0.5, \nu^- = 1$.

	u_1		u_2		p	
N	$\|e\|_{L^2}$	$\|e\|_{H^1}$	$\|e\|_{L^2}$	$\|e\|_{H^1}$	$\|e\|_{L^2}$	$\|e\|_{H^1}$
6	1.1024e-02	1.4136e-01	1.7768e-02	1.0353e-01	1.3103e-01	1.9335e+00
10	1.3774e-04	1.5536e-03	1.3028e-04	1.0648e-03	1.9761e-03	7.6296e-02
14	3.8301e-05	5.0922e-04	2.9098e-05	3.2607e-04	6.1366e-04	1.7891e-03
18	1.7160e-07	1.2512e-06	1.6626e-07	1.1818e-06	2.2675e-06	2.4578e-04
22	5.0688e-09	1.6974e-08	4.8894e-09	1.7565e-08	5.8622e-07	2.6011e-05
26	1.4857e-10	4.1623e-10	1.4310e-10	4.2503e-10	8.0753e-10	2.2968e-07
30	3.9334e-12	1.3770e-11	3.7632e-12	1.3742e-11	5.4204e-10	2.7457e-08

TABLE 6. Error discretization of Example 3 for Chebychev case, with $\nu^+ = 0.5, \nu^- = 1$.

	u_1		u_2		p	
N	$\|e\|_{L^2}$	$\|e\|_{H^1}$	$\|e\|_{L^2}$	$\|e\|_{H^1}$	$\|e\|_{L^2}$	$\|e\|_{H^1}$
6	1.5591e-02	2.5490e-01	1.7150e-02	1.9448e-01	4.6610e-01	4.6766e+00
10	1.0044e-04	4.0661e-03	1.0708e-04	2.6273e-03	4.5261e-03	3.9413e-01
14	3.8558e-05	1.1569e-03	2.1640e-05	8.5232e-04	2.3950e-03	1.0950e-02
18	8.0707e-08	5.6921e-07	7.2241e-08	6.8517e-07	9.1179e-07	9.4338e-04
22	2.0965e-09	1.5632e-08	1.8385e-09	1.6902e-08	2.8407e-08	2.5045e-05
26	5.6011e-11	3.8507e-10	4.8409e-11	4.4236e-10	7.3627e-09	7.2421e-07
30	1.8581e-12	3.4346e-11	1.6809e-12	7.4142e-11	7.5522e-09	5.6789e-08

solution, approximate solution and the error of velocity u_2 for $N = 18$ of this example is plotted in FIGURE 5.

5. CONCLUDING REMARKS

In this paper, we proposed pseudo-spectral method for Stokes problem with discontinuous viscosity and singular source term. First, we derived interface conditions and Stokes equations defined on each sub-domain. The interface conditions are continuity of velocity and coupled interface condition for velocity and pressure. By using only these two interface conditions we derive spectral accuracy. Then we obtain a simple and efficient algorithm applying pseudo-spectral method to each equation. It is shown that the proposed method can be applied to

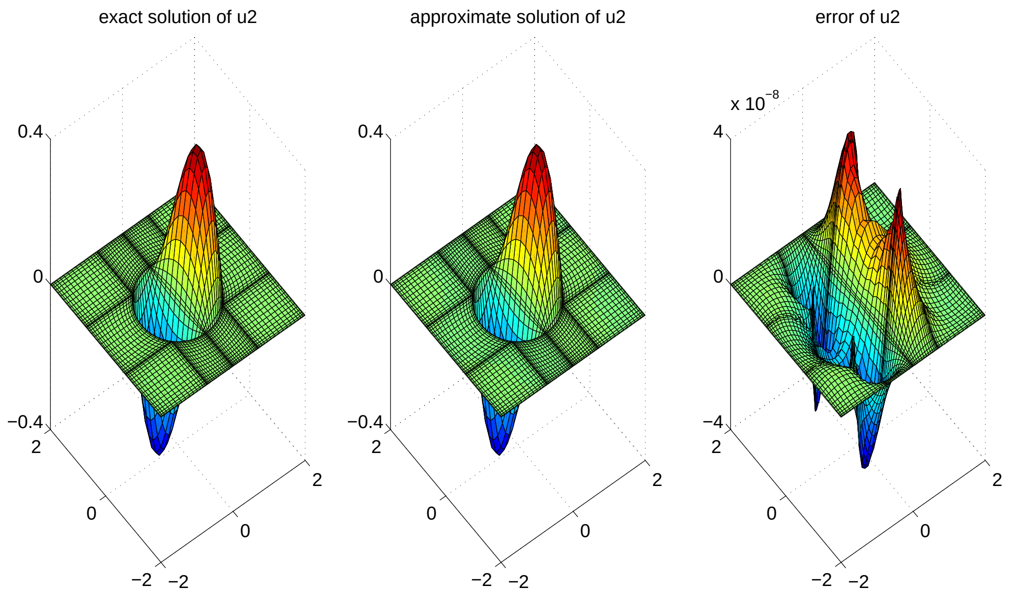

FIGURE 5. Exact and approximate solutions, and its error of velocity u_2 for $N = 18$ of Example 4.

TABLE 7. Error discretization of Example 4 for Legenre case, with $\nu^+ = 0.5, \nu^- = 1$.

	u_1		u_2		p	
N	$\|e\|_{L^2}$	$\|e\|_{H^1}$	$\|e\|_{L^2}$	$\|e\|_{H^1}$	$\|e\|_{L^2}$	$\|e\|_{H^1}$
6	3.1170e-03	1.9393e-02	2.1546e-03	1.8157e-02	1.7510e-02	2.3039e-01
10	8.5050e-05	4.9167e-04	7.1075e-05	4.1782e-04	6.9618e-04	5.6093e-02
14	7.0110e-07	1.0418e-05	1.0630e-06	8.4465e-06	1.2114e-05	1.7288e-03
18	2.1192e-08	1.2786e-07	3.4959e-08	1.6806e-07	1.5752e-07	2.3705e-05
22	2.1774e-09	3.5515e-08	3.2506e-09	3.6317e-08	5.4532e-08	1.5405e-05
26	4.0290e-11	4.1343e-10	4.5607e-11	4.7347e-10	1.0386e-09	1.8868e-07
30	1.4386e-12	5.2650e-11	2.3657e-12	4.8110e-11	8.2063e-09	2.3467e-08

Stokes interface problem defined on curved domain by using Gordon and Hall transformation. This means that the method can be extended to pseudo-spectral element method to solve Stokes interface problems defined on complicated domain. The numerical experiments also showed

TABLE 8. Error discretization of Example 4 for Chebychev case, with $\nu^+ = 0.5, \nu^- = 1$.

	u_1		u_2		p	
N	$\|e\|_{L^2}$	$\|e\|_{H^1}$	$\|e\|_{L^2}$	$\|e\|_{H^1}$	$\|e\|_{L^2}$	$\|e\|_{H^1}$
6	4.0280e-03	3.6396e-02	2.6758e-03	3.8719e-02	2.6615e-02	4.8940e-01
10	2.2951e-05	3.6981e-04	2.9940e-05	4.0711e-04	3.8221e-04	5.9888e-02
14	4.7602e-07	4.1613e-05	6.9575e-07	3.0613e-05	5.1888e-05	9.5297e-03
18	9.9876e-09	2.9337e-07	1.6508e-08	3.5445e-07	4.3810e-07	1.0009e-04
22	2.3777e-10	7.5238e-09	3.4763e-10	7.8269e-09	9.6076e-09	3.0175e-06
26	7.1970e-12	4.9720e-10	1.0349e-11	5.4790e-10	1.6688e-09	2.3394e-07
30	5.7495e-13	5.4610e-11	7.3574e-13	1.0336e-10	1.0196e-08	3.5471e-08

that the method has the spectral convergence high accuracy. Furthermore the method can be adopted to solve other interface problems such as Navier-Stokes interface problems.

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