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THE NOVELTY OF INFINITE SERIES FOR THE COMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND

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ABSTRACT. According to the fact that the low convergence level of the complete elliptic integral of the first kind for the modulus which having values approach to one. In this paper we propose novelty of the complete elliptic integral which having new infinite series that consists of new modulus introduced as own modulus function. We apply scheme of iteration by substituting the common modulus with own modulus function into the new infinite series. We obtained so many new exact formulas of the complete elliptic integral derived from this method correspond to the number of iterations. On the other hand, it has been also obtained a lot of new transformation functions with the corresponding own modulus functions. The calculation results show that the enhancement of the number of significant figures of the new infinite series of the complete elliptic integral of the first kind corresponds to the level of quadratic convergence.

1. INTRODUCTION

The complete elliptic integral of the first kind $K(k)$ is one of three elliptic integrals that getting a lot of attentions. It is not only used by mathematicians but also by engineers. On the development of scientifics for instance, the complete elliptic integral of the first kind was commonly used in studying a wide variety of problems involving three dimensional lattices [1], for creating Pi formula via Arithmetic Geometric Mean [2, 3], for building analytical solution of the nonlinear pendulum [4], as the basis for generalizing incomplete elliptic integral of the first kind [5], as the basis of development hypergeometric series [6], etc. Whereas in the fields of application, it was widely used in the design of electromagnetic devices, namely as basic function in conformal mapping which is mathematical tool for solving electromagnetic problems [7, 8, 9], as mathematical model for designing parallel plate capacitor [10], curved patch capacitor [11], and microstrip [12, 13] that were encountered in the fields of communication

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especially for antennas application and detectors. The first kind $K(k)$ can be used to obtain the complete elliptic integral of the second kind $E(k)$ because both of these functions having relationship of ordinary differential equation [14], and Legendre relation [15]. However, $K(k)$ can be calculated in several ways, that are by using power series, Fourier series, theta functions, and Landen transformation. The first three methods are only convenient and useful for small k (approaching zero), unfortunately they are not convergent for the value of large k (approaching one). On the other hand, the Landen transformation is rapidly convergent, but are non-trivial to be applied [16]. Therefore the enhancement of convergence level of the $K(k)$ which consisting large k remains interesting to be considered.

In this early assignment, we focus to enhance the convergence level of the complete elliptic integral of the first kind $K(k)$ by transforming the value of modulus k into an appropriate modulus functions to produce transformation functions. From the literature review that we have conducted, there are two well known examples of such modulus transformation, namely $k \to ik/k'$ and $k \to (1-k')/(1+k')$ that are as the generating transformation functions of $K(k) = \frac{1}{k'}K\left(ik/k'\right)$ and $K(k) = \frac{2}{1+k'}K((1-k')/(1+k'))$ respectively, in which $i = \sqrt{-1}$, and $k' = \sqrt{1-k^2}$ is the complementary of modulus k [17]. Nevertheless, it is necessarry to find the other forms of transformation function that provide higher degree of convergence level. For this purpose, we perform modification to the original integral form of $K(k)$ to obtain new form of its infinite series. Further, from this new infinite series will be known the new transformation function of the elliptic integral and the corresponding modulus function. The modulus function of k will be useful to enhance the level of $K(k)$ convergence through employing the other scheme of iteration beyond that has been applied on previous work as mentioned in Borwein's book [18].

2. EQUATIONS AND THEOREMS

2.1. **Formulation Of The New Infinite Series Of The Complete Elliptic Integral Of The First Kind.** In order to obtain the new infinite series version of the $K(k)$, we firstly recall the following definition of the complete elliptic integral of the first kind that available in so many text books of mathematics, for instance in Carlson [17], Borwein [18], and Boas [19],

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \ ,k \in (0,1). \tag{2.1}$$

We call (2.1) as the original complete elliptic integral of the first kind that after expanding $(1-k^2\ \sin\theta)^{-1/2}$ and integrating term by term, we obtained the following infinite series,

$$K(k) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{3}{8}\right)^2 k^4 + \left(\frac{5}{16}\right)^2 k^6 + \left(\frac{35}{128}\right)^2 k^8 + \cdots\right], \tag{2.2}$$

in which the three dots means continuing on indefinitely. The infinite series of $K(k)$ corresponds to the form,

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{2^n\ n!}\right]^2 k^{2n}. \tag{2.3}$$

The fact that the double factorial of $(2n-1)$ can be represented as following,

$$(2n-1)!! = \frac{(2n)!}{2^n\, n!}, \tag{2.4}$$

then the infinite series in (2.3) can be written in the following form,

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}\, n!^2}\right]^2 k^{2n}. \tag{2.5}$$

In formulating the new version of the $K(k)$ infinite series, we firstly modify the integral form in (2.1) by varying the angle θ into the double angle 2θ through the relationship of the following trigonometric identity,

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta). \tag{2.6}$$

Substituting (2.6) into (2.1) gives,

$$K(k)_N = \frac{1}{\sqrt{1-\frac{k^2}{2}}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{\cos 2\theta}{1-2/k^2}}}. \tag{2.7}$$

where the subscript N is included to distinguish the new integral from its original form. The infinite series of the new elliptic integral of (2.7) is obtained in the form,

$$K(k)_N = \frac{\pi}{2\sqrt{1-k^2/2}}\sum_{n=0}^{\infty}\frac{(4n)!}{(2^{3n}n!)^2(2n)!}\left(\frac{1}{1-2/k^2}\right)^{2n}, \tag{2.8}$$

or it can be written as,

$$K(k)_N = \frac{\pi}{2\sqrt{1-k^2/2}}\sum_{n=0}^{\infty}\frac{(4n-1)!}{(2^{2n}n!)^2}\left(\frac{1}{1-2/k^2}\right)^{2n}, \tag{2.9}$$

On both (2.8) and (2.9), we have employed the following relationship of (2.4) by replacing with , namely,

$$(4n-1)!! = \frac{(4n)!}{2^n\,(2n)!}, \tag{2.10}$$

Other form of the new complete elliptic integral of the first kind is of form,

$$K(k)_N = \frac{1}{\sqrt{1-k^2/2}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1+0.5\left(k\Big/\sqrt{1-k^2/2}\right)^2\cos 2\theta}}, \tag{2.11}$$

which having infinite series in the form,

$$K(k)_N = \frac{\pi}{2\sqrt{1-k^2/2}}\sum_{n=0}^{\infty}\frac{(4n-1)!!}{(2^{3n}n!)^2}\left(\frac{k}{\sqrt{1-k^2/2}}\right)^{4n}, \tag{2.12}$$

where for the first four terms as following,

$$K(k)_N = \frac{\pi}{2\sqrt{1-k^2/2}} \left\{ 1 + \frac{1.3}{8^2}\left(\frac{k}{\sqrt{1-k^2/2}}\right)^4 + \frac{1.3.5.7}{128^2}\left(\frac{k}{\sqrt{1-k^2/2}}\right)^8 \right. \\ \left. + \frac{1.3.5.7.9.11}{3072^2}\left(\frac{k}{\sqrt{1-k^2/2}}\right)^{12} + \cdots \right\} \tag{2.13}$$

which can further be simplified to the form,

$$K(k)_N = \frac{\pi}{2\sqrt{1-k^2/2}} \left\{ 1 + \frac{1}{2}\frac{3}{8}\left(\frac{1}{\sqrt{1-2/k^2}}\right)^2 + \frac{3}{8}\frac{35}{128}\left(\frac{1}{\sqrt{1-2/k^2}}\right)^4 \right. \\ \left. + \frac{5}{16}\frac{231}{1024}\left(\frac{1}{\sqrt{1-2/k^2}}\right)^6 + \cdots \right\} \tag{2.14}$$

It appears that (2.14) is the expansion of (2.8) and/or (2.9).

2.2. **Formulation Of New Transformation Function For The Complete Elliptic Integral Of The First Kind.** Before performing the step of formulation for finding the new transformation function of $K(k)$ and/or $K(k)_N$, it is necessary to show that really both original and new version of the complete elliptic integral of first kind are equal. Both integrals are only different in the convergence level of its infinite series. Of course $K(k)_N$ will reduce to $K(k)$ when 2θ is varied back into θ. Nevertheless, because $\cos 2\theta$ has two definitions, then varying the cosine of $\cos 2\theta$ must be performed one by one of each definition. Beginning by introducing the following variable,

$$x = \frac{1}{\sqrt{1-k^2/2}}, \tag{2.15}$$

so that $K(k)_N$ in (2.11) can be written in the form,

$$K(k)_N = x\int_0^{\pi/2} \frac{d\theta}{\sqrt{1+0.5(kx)^2\cos 2\theta}}. \tag{2.16}$$

Further, into (2.16) we firstly subtitute the following cosine of 2θ ,

$$\cos 2\theta = 1 - 2\sin^2\theta, \tag{2.17}$$

which giving the following integral form,

$$K(k)_N = A\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k_{1N}^2\sin^2\theta}}, \tag{2.18}$$

where

$$A = \frac{x}{\sqrt{1+0.5(kx)^2}} = \frac{1}{\sqrt{1-k^2/2}} \frac{1}{\sqrt{1+\frac{k^2}{2}\frac{1}{1-k^2/2}}} = 1, \tag{2.19}$$

and

$$k_{1N} = \frac{kx}{\sqrt{1+0.5(kx)^2}} = k. \tag{2.20}$$

With the above values of $A = 1$ and $k_{1N} = k$, it appears that (2.18) has verified the equality of

$$K(k)_N = K(k). \tag{2.21}$$

After using the following cosine of 2θ

$$\cos 2\theta = 2\cos^2\theta - 1, \tag{2.22}$$

then $K(k)_N$ in (2.11) can be written as,

$$K(k)_N = B\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k_{2N}^2\cos^2\theta}}, \tag{2.23}$$

where,

$$B = \frac{x}{\sqrt{1-0.5(kx)^2}} = \frac{1}{\sqrt{1-k^2/2}} \frac{1}{\sqrt{1-\frac{k^2}{2}\frac{1}{1-k^2/2}}} = \frac{1}{\sqrt{1-k^2}} = \frac{1}{k'}, \tag{2.24}$$

and

$$k_{2N} = \frac{ikx}{\sqrt{1-0.5(kx)^2}} = \frac{ik}{k'}, \tag{2.25}$$

Further, by using both values of B and k_{2N} above then (2.23) becomes,

$$K(k)_N = \frac{1}{k'}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-(ik/k')^2\cos^2\theta}}. \tag{2.26}$$

Equation (2.26) indicates that there is the other form of the complete elliptic integral of first kind $K(k)$ than the original form on (2.1), namely in the form,

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\cos^2\theta}}. \tag{2.27}$$

Further, by involving the new definition of $K(k)$, then from (2.26) we obtain the following transformation function,

$$K(k)_N = \frac{1}{k'}K\left(\frac{ik}{k'}\right). \tag{2.28}$$

Due to the equality of (2.21), then from (2.28) it can also be formed the following transformation function,

$$K(k) = \frac{1}{k'}K\left(\frac{ik}{k'}\right). \tag{2.29}$$

Ones can also verify (2.21) and (2.27) form (2.11). Applying $\cos 2\theta$ from the (2.17) gives transformation function in (2.21), while applying $\cos 2\theta$ from (2.22) produces transformation function in (2.28). The equality of $K(k)_N$ and $K(k)$ also presents the equality of transformation function in (2.29) that giving,

$$K(k)_N = \frac{1}{k'} K\left(\frac{ik}{k'}\right)_N . \tag{2.30}$$

Also, the following transformation function,

$$K(k) = \frac{2}{1+k'} K\left(\frac{1-k'}{1+k'}\right), \tag{2.31}$$

gives,

$$K(k)_N = \frac{2}{1+k'} K\left(\frac{1-k'}{1+k'}\right)_N , \tag{2.32}$$

Nevertheless, due to both infinite series $K(k)$ and $K(k)_N$ are different then the convergence level of (2.29) and (2.31) are also different with (2.30) and (2.32), respectively. In order to obtain the new transformation function of $K(k)_N$, we explore the right side of (2.32) by exerting the change of modulus $k \to (1-k')/(1+k')$ into $K(k)_N$ in (2.7), so we find,

$$K\left(\frac{1-k'}{1+k'}\right)_N = \frac{\sqrt{2}(1+k')}{\sqrt{2(1+k')^2-(1-k')^2}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{(1-k')^2\cos 2\theta}{(1-k')^2-2(1+k')^2}}}, \tag{2.33}$$

after applying the following identity,

$$(1+k')^2 = 4k' + (1-k')^2, \tag{2.34}$$

then (2.33) becomes,

$$K\left(\frac{1-k'}{1+k'}\right)_N = \frac{\sqrt{2}(1+k')}{\sqrt{(1+k')^2+4k'}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1+\frac{(1-k')^2\cos 2\theta}{(1-k')^2+4k'}}}. \tag{2.35}$$

In addition, applying the cosine of 2θ from (2.17), then (2.35) can be simplified as,

$$K\left(\frac{1-k'}{1+k'}\right)_N = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-((1-k')/(1+k'))^2\sin^2\theta}}. \tag{2.36}$$

As previous explanation, from (2.36) appears that applying the cosine of 2θ as in (2.21) only gives an equality, i.e.,

$$K\left(\frac{1-k'}{1+k'}\right)_N = K\left(\frac{1-k'}{1+k'}\right). \tag{2.37}$$

while applying the cosine of 2θ from (2.22) gives,

$$K\left(\frac{1-k'}{1+k'}\right)_N = \frac{1+k'}{2\sqrt{k'}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\left((1-k)/(2i\sqrt{k'})\right)^2\cos^2\theta}}, \tag{2.38}$$

while applying the cosine of 2θ from (2.22) gives,

$$K\left(\frac{1-k'}{1+k'}\right)_N = \frac{1+k'}{2\sqrt{k'}}K\left(\frac{1-k'}{2i\sqrt{k'}}\right). \tag{2.39}$$

Here we introduce $\frac{1-k'}{2i\sqrt{k'}}$ as own modulus function. By considering (2.32), then we obtain,

$$K(k)_N = \frac{1}{\sqrt{k'}}K\left(\frac{1-k'}{2i\sqrt{k'}}\right). \tag{2.40}$$

Finally, we obtain a new transformation function in the following form,

$$K(k)_N = \frac{1}{\sqrt{k'}}K\left(\frac{1-k'}{2i\sqrt{k'}}\right)_N. \tag{2.41}$$

2.3. **Enhancement The Level Of Convergence Of The Complete Elliptic Integral Of The First Kind By Applying The Scheme Of Iteration To Its New Transformation Function.** As mentioned previously that the infinite series of the complete elliptic integral of the first kind is slowly convergence. To enhance the level of convergence, we implement the scheme of iteration to the transformation functions of $K(k)$. Here, we just involving two transformation functions in (2.30) and (2.41). Starting with (2.30), after exerting the change of modulus $k \to ik/k'$ into (2.8) to forms $K\left(ik/k'\right)_N$ so we obtain,

$$K_1(k)_N = \frac{1}{k'}\frac{\pi}{2\sqrt{1-\frac{1}{2}\left(\frac{ik}{k'}\right)^2}}\sum_{n=0}^{\infty}\frac{(4n)!}{(2^{3n}n!)^2(2n)!}\left(\frac{1}{1-2\left(\frac{k'}{ik}\right)^2}\right)^{2n}, \tag{2.42}$$

substituting the complementary modulus $k' = \sqrt{1-k^2}$ into (2.42), then we have,

$$K_1(k)_N = \frac{\sqrt{2}\pi}{2\sqrt{2-k^2}}\sum_{n=0}^{\infty}\frac{(4n)!}{(2^{3n}n!)^2(2n)!}\left(\frac{k^2}{2-k^2}\right)^{2n}. \tag{2.43}$$

Due to (2.43) can reduce to (2.8), we conclude that the scheme of iteration by the change modulus $k \to \frac{ik}{k'}$ can not be used to enhance the level of convergence of the complete elliptic integral of the first kind. Therefore, the implemetation of the iteration scheme is now focused on (2.41),

$$K_m(k)_N = \frac{1}{\sqrt{k'}}K_{m-1}\left(\frac{1-k'}{2i\sqrt{k'}}\right)_N, \quad m = 1, 2, 3... \tag{2.44}$$

here m is the step of iteration, whereas $K_0(k)_N$ is the infinite series of the new version of elliptic integral in (2.8) and/or (2.9). But for simplicity we choose the form of infinite series of (2.9), where for the first iteration $(m = 1)$, we obtain

$$K_1(k)_N = \frac{1}{\sqrt{k'}}K_0\left(\frac{1-k'}{2i\sqrt{k'}}\right)_N. \tag{2.45}$$

After exerting the change of modulus $k \to \frac{1-k'}{2i\sqrt{k'}}$ into (2.9), we obtain the following infinite series of $K_1(k)$, namely:

$$K_1(k)_N = \frac{\sqrt{2}\pi}{\sqrt{1+k'^2+6k'}} \sum_{n=0}^{\infty} \frac{(4n-1)!!}{(2^{2n}n!)^2} \left(\frac{(\sqrt{1-k'})^2}{\sqrt{1+k'^2+6k'}} \right)^{4n}. \tag{2.46}$$

Further, for the second iteration $(m = 2)$ we obtain,

$$K_2(k)_N = \frac{1}{\sqrt{k'}} K_1 \left(\frac{1-k'}{2i\sqrt{k'}} \right)_N. \tag{2.47}$$

However, before applying the change of modulus $k \to \frac{1-k'}{2i\sqrt{k'}}$ into $K_1(k)_N$ on (2.46), we must substitute $k' = \sqrt{1-k^2}$ so that (2.47) forms the following infinite series, namely,

$$K_2(k)_N = \frac{1}{\sqrt{k'}} \frac{\sqrt{2}\pi}{\sqrt{2-k^2+6\sqrt{1-k^2}}} \sum_{n=0}^{\infty} \frac{(4n-1)!!}{(2^{2n}n!)^2} \left(\frac{1-\sqrt{1-k^2}}{\sqrt{2-k^2+6\sqrt{1-k^2}}} \right)^{4n}. \tag{2.48}$$

Finally, the change of modulus $k \to \frac{1-k'}{2i\sqrt{k'}}$ into (2.48) gives,

$$\begin{aligned} K_2(k)_N =& \frac{2\sqrt{2}\pi}{\sqrt{1+k'^2+6k'+6(1+k')\sqrt{4k'}}} \\ &\times \sum_{n=0}^{\infty} \frac{(4n-1)!!}{(2^{2n}n!)^2} \left(\frac{(1-\sqrt{k'})^2}{\sqrt{1+k'^2+6k'+6(1+k')\sqrt{4k'}}} \right)^{4n}. \end{aligned} \tag{2.49}$$

The same procedure to the second iteration, for the third iteration $(m = 3)$,

$$K_3(k)_N = \frac{1}{\sqrt{k'}} K_2 \left(\frac{1-k'}{2i\sqrt{k'}} \right)_N. \tag{2.50}$$

we obtain the following infinite series,

$$\begin{aligned} K_3(k)_N =& \frac{4\sqrt{2}\pi}{\sqrt{1+k'^2+6k'+6(1+k')\sqrt{4k'}+6(1+\sqrt{k'})^2\sqrt{4(1+k')\sqrt{4k'}}}} \\ &\times \sum_{n=0}^{\infty} \frac{(4n-1)!!}{(2^{2n}n!)^2} \left(\frac{(\sqrt{1+k'}-\sqrt[4]{4k'})^2}{\sqrt{1+k'^2+6k'+6(1+k')\sqrt{4k'}+6(1+\sqrt{k'})^2\sqrt{4(1+k')\sqrt{4k'}}}} \right)^{4n}. \end{aligned} \tag{2.51}$$

By similarly way for the fourth iteration $(m = 4)$, we obtain

$$K_4(k)_N = \frac{1}{\sqrt{k'}} K_3 \left(\frac{1-k'}{2i\sqrt{k'}} \right)_N. \tag{2.52}$$

which giving the following infinite series,

$$K_4(k)_N = \frac{8\sqrt{2}\pi}{\sqrt{\begin{aligned}&1+k'^2+6k'+6(1+k')\sqrt{4k'}+6(1+\sqrt{k'})^2\sqrt{4(1+k')\sqrt{4k'}}\\&+12(\sqrt{1+k'}+\sqrt[4]{4k'})^2(1+\sqrt{k'})\sqrt[4]{4(1+k')\sqrt{4k'}}\end{aligned}}}$$
$$\times\sum_{n=0}^{\infty}\frac{(4n-1)!!}{(2^{2n}n!)^2}\frac{\left(1+\sqrt{k'}-\sqrt[4]{4(1+k')\sqrt{4k'}}\right)^2}{\sqrt{\begin{aligned}&1+k'^2+6k'+6(1+k')\sqrt{4k'}+6(1+\sqrt{k'})^2\sqrt{4(1+k')\sqrt{4k'}}\\&+12(\sqrt{1+k'}+\sqrt[4]{4k'})^2(1+\sqrt{k'})\sqrt[4]{4(1+k')\sqrt{4k'}}\end{aligned}}}. \tag{2.53}$$

After performing the simplification of algebra processes, the first four exact formulas of $K_m(k)_N$ infinite series above can be expressed in each transformation function, namely,

$$K_1(k)_N = \frac{1}{k'}K(k_1)_N,\ \ k_1 = \frac{1-k'}{2i\sqrt{k'}} \tag{2.54}$$

$$K_2(k)_N = \frac{2K(k_2)_N}{\sqrt{(1+k')\sqrt{4k'}}},\ \ k_2 = \frac{(1-\sqrt{k'})^2}{2i\sqrt{(1+k')\sqrt{4k'}}} \tag{2.55}$$

$$K_3(k)_N = \frac{4K(k_3)_N}{(1+\sqrt{k'})\sqrt[4]{4(1+k')\sqrt{4k'}}},\ \ k_3 = \frac{(\sqrt{1+k'}-\sqrt[4]{4k'})^2}{2i(1+\sqrt{k'})\sqrt[4]{4(1+k')\sqrt{4k'}}} \tag{2.56}$$

and

$$\begin{aligned}K_4(k)_N &= \frac{8K(k_4)_N}{\left(\sqrt{1+k'}+\sqrt[4]{4k'}\right)\sqrt{2(1+\sqrt{k'})\sqrt[4]{(1+k')\sqrt{4k'}}}},\\ k_4 &= \frac{\left(1+\sqrt{k'}-\sqrt[4]{4(1+k')\sqrt{4k'}}\right)^2}{2i\left(\sqrt{1+k'}+\sqrt[4]{4k'}\right)\sqrt{2(1+\sqrt{k'})\sqrt[4]{(1+k')\sqrt{4k'}}}}\end{aligned} \tag{2.57}$$

where k_1 , k_2 ,k_3 , and k_4 are the corresponding own modulus functions of first four iterations.

3. Result and Discussion

The discussion about the enhance of the convergence level of the complete elliptic integral of the first kind here is focused to give some comments to achievement of significant figures of both original and new infinite series. All calculations were performed by using the facilities of integral, summation, and evaluation of function that available on MapleV-Soft. Beginning by presenting the calculations results of the significant figures of infinite series of the original $K(k)$ in (2.2) as shown in Table 1. Here, l denotes the highest term in each infinite

TABLE 1. Significant figures of infinite series of the original for the number of terms multiple of ten

l	k=1/10	k=9/10
0	1.570796326…	1.570796326…
10	1.574745562…	2.262667579…
20	1.574745562…	2.279280028…
30	1.574745562…	2.280439683…
40	1.574745562…	2.280538812…

series of $K(l)$ After comparing the numerical values of the original integral form in (2.1) i.e., $K(1/10) = 1.574745562 \quad 1517356\ldots$ and $K(9/10) = 2.28054913 \quad 8422770\ldots$, the number of significant figures for the modulus $K(9/10)$ that are too little and slow for the number of terms multiple of ten comparing with the achievement of $k = 1/10$. It has verified the statement in [16] that power series of the complete elliptic integral of the first kind is slowly convergent for the value of modulus k approaches one. Further, to verify our statement above that really the exact values of the original elliptic integral in (2.1) and both of its new version in (2.7) and (2.11) are equal, we present the results of calculation in Table 2 below. Here we truncate numerical value of all calculations up to 16 significant figures. However, as shown in

TABLE 2. The exact value of the original and new version of the complete elliptic integral of the first kind.

k	$K(k)$	$K(k)_N$
1/10	1.574745561517356…	1.574745561517356…
1/2	1.685750354812596…	1.685750354812596…
$1/\sqrt{2}$	1.854074677301372…	1.854074677301372…
9/10	2.280549138422770…	2.280549138422770…

the following Table 3, the numerical values of both infinite series $K(k)$ and $K(k)_N$ are still different. Although to reach 16 significant figures are still required so many terms, but it appears that for all of modulus the number of terms required by the $K(k)_N$ are more little. This fact as a guarantee that the new version of the complete elliptic integral of the first kind is faster to converge compared to the original one. The enhancement convergence level of the complete elliptic integral of the first kind can be traced by considering the significant figures resulted for each highest term of the original version of the complete integral $K(k)$ on (2.5), $K(k)_N$ of the new version on (2.12), and the iterative version $K_1(k)_N$ on (2.42). The calculation results for the values of modulus 1/10.$1/\sqrt{2}$, and 9/10 can be seen in Table 4 below, The results of calculation for the three values of modulus k ranging from small until big values as shown in

TABLE 3. Highest term l of $K(k)$ and $K(k)_N$ infinite series to reach 16 significant figures

k	l of $K(k)$	l of $K(k)_N$
1/10	6	4
1/2	24	8
$1/\sqrt{2}$	45	14
9/10	150	41

TABLE 4. Comparison of Significant figures of the first six terms of the original, new, and first iteration of the complete elliptic integral of the first kind

l	$K(k)$			$K(k)_N$			$K_1(k)_N$		
	$k=\frac{1}{10}$	$k=\frac{1}{\sqrt{2}}$	$k=\frac{9}{10}$	$k=\frac{1}{10}$	$k=\frac{1}{\sqrt{2}}$	$k=\frac{9}{10}$	$k=\frac{1}{10}$	$k=\frac{1}{\sqrt{2}}$	$k=\frac{9}{10}$
0	3	1	0	5	2	1	12	3	2
1	5	1	0	10	3	2	23	8	5
2	7	2	1	15	3	2	29	10	7
3	9	2	1	20	6	2	41	15	8
4	11	2	1	24	6	2	55	19	10
5	13	3	1	30	7	2	67	23	12

Table 4 confirms again that the significant figures of the new version of the complete elliptic integral of the first kind are more than the significant figures of the original integral form. For closing this discussion, we present the sequence approximation formulas obtained by setting the highest of term $l=0$ into all of new infinite series formulas in (2.46), (2.49), (2.51), and (2.53), namely:

$$K_{1,0}(k)_N = \frac{\pi\sqrt{2}}{\sqrt{1+k'^2+6k'}}, \tag{3.1}$$

$$K_{2,0}(k)_N = \frac{2\sqrt{2}\pi}{\sqrt{1+k'^2+6k'+6(1+k')\sqrt{4k'}}}, \tag{3.2}$$

$$K_{3,0}(k)_N = \frac{4\sqrt{2}\pi}{\sqrt{1+k'^2+6k'+6(1+k')\sqrt{4k'}+6(1+\sqrt{k'})^2\sqrt{4(1+k')\sqrt{4k'}}}}, \quad (3.3)$$

and

$$K_{4,0}(k)_N = \frac{8\sqrt{2}\pi}{\sqrt{\begin{array}{l}1+k'^2+6k'+6(1+k')\sqrt{4k'}+6(1+\sqrt{k'})^2\sqrt{4(1+k')\sqrt{4k'}} \\ \quad +12(\sqrt{1+k'}+\sqrt[4]{4k'})^2(1+\sqrt{k'})\sqrt[4]{4(1+k')\sqrt{4k'}}\end{array}}}, \quad (3.4)$$

By applying the same iteration scheme of (2.44), but here we replace $K_{m-1}\left(\frac{1-k'}{2i\sqrt{k'}}\right)_N$ with $K_{m-1}\left(\frac{1-k'}{2i\sqrt{k'}}\right)$, where $K_m(k)$ is the infinite series of the original complete elliptic integral in (2.5). The sequences of approximation formulas for the first term of $K_m(k)$ are obtained in the following forms,

$$K_{1,0}(k)_N = \frac{\pi}{2\sqrt{k'}}, \quad (3.5)$$

$$K_{2,0}(k)_N = \frac{\pi}{\sqrt{(1+k')\sqrt{4k'}}}, \quad (3.6)$$

$$K_{3,0}(k)_N = \frac{2\pi}{(1+\sqrt{k'})\sqrt{2(1+k')\sqrt{4k'}}}, \quad (3.7)$$

$$K_{4,0}(k)_N = \frac{4\pi}{\left(\sqrt{1+k'}+\sqrt[4]{4k'}\right)\sqrt{2(1+\sqrt{k'})\sqrt[4]{4(1+k')\sqrt{4k'}}}}, \quad (3.8)$$

On all sequences of approximation formulas $K_{m,l}(k)_N$ in (3.1)-(3.4) and $K_{m,l}(k)$ in (3.5)-(3.8), we have put the subscript l to indicate the highest term used in each infinite series. As previously, we set $l=0$ which means that all of the successive formulas contain only one term. Finally we present the comparison of calculation results in Table 5,

In the Table 4, although the number of significant figures for all of modulus k increase with increasing the number of terms, however we can not specify how much the number enhancement of such significant figures. But from the significant figures of the sequences of approximation formulas of the first term of $K_{m,l}(k)_N$ as shown in Table 5, it can be known that the ratio between the number of significant figures of two successive approximation formulas is approximately 2, that also holds for $K_{m,l}(k)$. Here, it means that the enhancement of convergence level of the complete elliptic integral by applying both iteration schemes $K_m(k)_N = \frac{1}{\sqrt{k'}}K_{m-1}\left(\frac{1-k'}{2i\sqrt{k'}}\right)_N$ for the new complete elliptic integral, and $K_m(k) = \frac{1}{\sqrt{k'}}K_{m-1}\left(\frac{1-k'}{2i\sqrt{k'}}\right)$ for the original integral form correspond to the level of quadratic convergence. However, the fact that the number of significant figures of $K_{m,0}(k)_N$ that

TABLE 5. Significant figures of the $K_{m,l}(k)_N$ and $K_{m,l}(k)$ for the first term $(l = 0)$

l	$m = 1$		$m = 2$		$m = 3$		$m = 4$	
	$K_{1,0}(k)_N$	$K_{1,0}(k)$	$K_{2,0}(k)_N$	$K_{2,0}(k)$	$K_{3,0}(k)_N$	$K_{3,0}(k)$	$K_{4,0}(k)_N$	$K_{4,0}(k)$
1/10	12	6	25	12	50	24	101	51
1/2	5	3	13	6	27	13	54	27
$1/\sqrt{2}$	3	2	10	4	21	11	43	21
9/10	2	1	6	3	13	7	30	15

always twice than $K_{m,0}(k)$ as shown in Table 5 indicates that the infinite series of the complete elliptic integral is faster to converge than the original one. This is related to the utility of the angle argument in the definition of complete elliptic integrals of the first kind, where the expression of the double angle 2θ gives higher convergence level than the angle θ.

4. CONCLUSIONS

From explanation and discussion above we take several conclusions. The complete elliptic integral of the first kind can be modified into the new form by varying the argument of angle θ into the double angle 2θ. Applying the scheme of iteration by substituting the common modulus k with the own modulus function $\frac{1-k'}{2i\sqrt{k'}}$ into the new infinite series produces so many new exact formulas of the complete elliptic integral correspond to the number of iterations. On the other hand, from the new transformation functions have been also obtained a lot of new transformation functions with the corresponding new modulus functions. The calculation results show that the enhancement of the number of significant figures of the new infinite series of the complete elliptic integral of the first kind corresponds to the level of quadratic convergence.

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