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A POSTERIORI ERROR ESTIMATORS FOR THE STABILIZED LOW-ORDER FINITE ELEMENT DISCRETIZATION OF THE STOKES EQUATIONS BASED ON LOCAL PROBLEMS

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ABSTRACT. In this paper we propose and analyze two a posteriori error estimators for the stabilized P_1/P_1 finite element discretization of the Stokes equations. These error estimators are computed by solving local Poisson or Stokes problems on elements of the underlying triangulation. We establish their asymptotic exactness with respect to the velocity error under certain conditions on the triangulation and the regularity of the exact solution.

1. INTRODUCTION

It is well known that a pair of finite element spaces for the velocity and the pressure of the Stokes equations cannot be chosen arbitrarily but should satisfy the discrete inf-sup condition (also known as the LBB condition). This condition prohibits the computationally convenient equal-order combination which uses the same element for both the velocity and the pressure. To remedy the situation for the low-order P_1/P_1 pair, Brezzi and Pitkäranta [1] added a weighted Laplace operator of the pressure to the continuity equation for the purpose of stabilization. Afterwards, several stabilized formulations were introduced in the late 1980's by adding the weighted residual of the momentum equation to the original formulation of the Stokes equations [2–5]. These formulations are consistently stabilized methods and allow any combination of velocity and pressure finite element spaces. In the equal-order P_1/P_1 case, they are reduced to the method of Brezzi and Pitkäranta [1] with a modified right-hand side. More recently, different approaches using the local pressure or pressure gradient projections have been studied by many authors; see, for example, [6–9].

One remarkable feature of the stabilized P_1/P_1 finite element discretization is that the a priori error estimate predicts the $O(h)$ convergence for the velocity in the H^1 norm and the pressure in the L^2 norm, but the $O(h^{3/2})$ convergence is numerically observed for the pressure in the L^2 norm. This observation was theoretically explained by the work of Eichel et al [10]

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where the $O(h^{3/2})$ superconvergence is established under certain conditions on the triangulation and the regularity of the exact solution.

A posteriori error estimation provides quantitative and/or qualitative information about the distribution of numerical errors and is indispensable for local error control and mesh refinement. There have been many works devoted to a posteriori error estimation for finite element discretizations of the Stokes equations employing the equal-order P_1/P_1 pair. In [11] Verfürth presented two error estimators for the linear part of the velocity approximation of the mini element, one of which is based on the residual of the finite element solution and the other one is based on solution of local Stokes problems. For stabilized P_1/P_1 finite element methods, Bank and Welfert [12] proposed some error estimators based on stabilized forms of local Stokes problems. Other types of error estimators based on the projection operator as well as the residual of the finite element solution were discussed in [13, 14]. We also refer to [15, 16] for analysis of the residual-based error estimator in the context of consistently stabilized finite element methods of general orders.

The goal of this paper is to propose and analyze two a posteriori error estimators for the stabilized P_1/P_1 finite element discretization of the Stokes equations. One is computed by solving local Poisson problems and the other one by solving local Stokes problems. Both local problems are adaptation of the ones considered by Kay and Silvester [17] for the stabilized P_1/P_0 finite element method. It is shown that our error estimators are equivalent to each other and locally efficient. Moreover, they are reliable under a certain saturation assumption. In particular, by virtue of the superconvergence result of [10], we will establish the asymptotic exactness of the velocity components of these error estimators with respect to the velocity error.

The rest of the paper is organized as follows. The next section describes the stabilized P_1/P_1 finite element discretization of the Stokes equations. In Sections 3 and 4 we define a posteriori error estimators based on local Poisson and Stokes problems and then establish their asymptotic exactness.

2. STABILIZED P_1/P_1 FINITE ELEMENT DISCRETIZATION

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with the boundary $\partial\Omega$. Given the vector-valued functions $\boldsymbol{f} \in (L^2(\Omega))^2$ and $\boldsymbol{g} \in (H^{1/2}(\partial\Omega))^2$, we consider the incompressible Stokes equations

$$\begin{cases} -\mu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\boldsymbol{u}$ and p represent the velocity and the pressure, respectively. The compatibility condition $\int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{n}\, ds = 0$, where $\boldsymbol{n}$ denotes the unit outward normal vector to $\partial\Omega$, is required to ensure existence and uniqueness of a solution $(\boldsymbol{u}, p)$.

The weak formulation for the problem (2.1) is to find $(\boldsymbol{u}, p) \in (H^1(\Omega))^2 \times L_0^2(\Omega)$ such that $\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{g}$ and

$$\begin{cases} \mu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_\Omega - (\nabla \cdot \boldsymbol{v}, p)_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega & \forall \boldsymbol{v} \in (H_0^1(\Omega))^2, \\ (\nabla \cdot \boldsymbol{u}, q)_\Omega = 0 & \forall q \in L_0^2(\Omega), \end{cases} \tag{2.2}$$

where $(\cdot,\cdot)_G$ (resp. $\langle\cdot,\cdot\rangle_{\partial G}$) is the standard L^2 inner product over a domain $G \subset \mathbb{R}^2$ (resp. over ∂G) and $L_0^2(\Omega)$ is the space of functions in $L^2(\Omega)$ with zero integral mean. From now on we set $\mu = 1$ and $\boldsymbol{g} = 0$ for simplicity.

In order to define a finite element discretization for (2.2), we introduce a family $\{\mathcal{T}_h\}_{h>0}$ of shape-regular conforming triangular meshes such that $\overline{\Omega} = \overline{\bigcup_{T \in \mathcal{T}_h} T}$ for each $h > 0$, where $h = \max_{T \in \mathcal{T}_h} h_T$ and h_T is the diameter of T.

Throughout the paper, C will denote a generic positive constant independent of the mesh size h which may be different at different places. We also denote the standard Sobolev norm and seminorm over a domain G by $\|\cdot\|_{s,p,G}$ and $|\cdot|_{s,p,G}$, respectively, with the convention that $\|\cdot\|_{s,2,G} = \|\cdot\|_{s,G}$ and $|\cdot|_{s,2,G} = |\cdot|_{s,G}$.

Let $P_r(T)$ be the space of all polynomials of degree $\leq r$ on T and let

$$W_h^r = \{v_h \in H^1(\Omega) : v_h|_T \in P_r(T) \ \ \forall T \in \mathcal{T}_h\}.$$

Then the velocity and pressure finite element spaces are chosen to be

$$\boldsymbol{V}_h = (W_h^1 \cap H_0^1(\Omega))^2, \qquad Q_h = W_h^1 \cap L_0^2(\Omega).$$

Since this pair does not satisfy the discrete inf-sup condition, Brezzi and Pitkäranta [1] considered the following stabilized form of (2.2): find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ such that

$$\begin{cases} (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h)_\Omega - (\nabla \cdot \boldsymbol{v}_h, p_h)_\Omega = (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ (\nabla \cdot \boldsymbol{u}_h, q_h)_\Omega + S_h(p_h, q_h) = 0 & \forall q_h \in Q_h, \end{cases} \tag{2.3}$$

where the stabilization term is given by

$$S_h(p, q) = \sum_{T \in \mathcal{T}_h} \gamma h_T^2 (\nabla p, \nabla q)_T$$

with a positive constant γ. We refer to [6–9] for the stabilization based on the local pressure or pressure gradient projections which does not involve the mesh parameter h_T or computation of derivatives.

Remark 1. *Due to the added stabilization term in (2.3), we have for* $q_h \in Q_h$

$$(\nabla \cdot \boldsymbol{u}, q_h)_\Omega + S_h(p, q_h) = S_h(p, q_h) \neq 0,$$

which means that the formulation (2.3) is inconsistent. To make it consistent, one may add the term $\sum_{T \in \mathcal{T}_h} \gamma h_T^2 (\boldsymbol{f}, \nabla q)_T$ *to the right-hand side of the second equation of (2.3) as is done in the residual-based stabilization [2, 3, 5].*

The following a priori error estimate for (2.3) can be found, for example, in [8]

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \le Ch(\|\boldsymbol{u}\|_{2,\Omega} + \|p\|_{1,\Omega}) \tag{2.4}$$

with respect to the the mesh-dependent norm

$$|||(\boldsymbol{v}, q)||| = \left(\|\nabla \boldsymbol{v}\|_{0,\Omega}^2 + \|q\|_{0,\Omega}^2 + \sum_{T\in\mathcal{T}_h} h_T^2 \|\nabla q\|_{0,T}^2\right)^{1/2}.$$

Finally, we present the superconvergence result for the method (2.3) proved in [10] when the triangulations $\{\mathcal{T}_h\}_{h>0}$ satisfy the following condition introduced in [18]:

Condition (α, σ): For each $h > 0$, the triangulation $\mathcal{T}_h$ can be partitioned into two disjoint sets $\mathcal{T}_{1,h} \cup \mathcal{T}_{2,h}$ with some positive constants α and σ in such a way that

- every two adjacent triangles of $\mathcal{T}_{1,h}$ form an $O(h^{1+\alpha})$ parallelogram, i.e., the lengths of any two opposite edges differ only by $O(h^{1+\alpha})$;
- the total area of $\bigcup_{T\in\mathcal{T}_{2,h}} T$ is $O(h^\sigma)$.

Roughly speaking, this condition means that most pairs of adjacent elements in $\mathcal{T}_h$ form almost parallelograms and there are only a small number of exceptional elements.

Suppose that the triangulations $\{\mathcal{T}_h\}_{h>0}$ satisfy the Condition (α, σ) and $(\boldsymbol{u}, p) \in (H^3(\Omega) \cap W^{2,\infty}(\Omega))^2 \times H^2(\Omega)$. Let $v_I \in W_h^1$ denote the standard nodal interpolant of $v \in C(\overline{\Omega})$. Then the following superconvergence result was established in [10]

$$|||(\boldsymbol{u}_I - \boldsymbol{u}_h, p - p_h)||| \le Ch^{1+\rho}(\|\boldsymbol{u}\|_{3,\Omega} + \|\boldsymbol{u}\|_{2,\infty,\Omega} + \|p\|_{2,\Omega}) \tag{2.5}$$

with $\rho = \min(\alpha, \frac{1}{2}, \frac{\sigma}{2})$. By comparing with (2.4), it should be observed that the pressure approximation is itself superconvergent. This explains the numerically observed $O(h^{3/2})$ convergence of the pressure on uniform meshes (with $\alpha = \sigma = \infty$).

3. Error Estimator Based on Local Poisson Problems

In this section we propose and analyze an error estimator for the stabilized P_1/P_1 finite element discretization (2.3) based on solution of local Poisson problems. This error estimator is an adaptation of the one considered by Kay and Silvester [17] for the stabilized P_1/P_0 finite element method.

Let us introduce some notation needed in defining the error estimator. The normal derivative of a vector-valued function $\boldsymbol{v}$ on ∂T is denoted by $\frac{\partial \boldsymbol{v}}{\partial n_T} := (\nabla \boldsymbol{v})\boldsymbol{n}_T$, where $\boldsymbol{n}_T$ is the unit normal outward to T, and its jump across an interior edge $e = \partial T \cap \partial T'$ is defined as

$$\left[\!\left[\frac{\partial \boldsymbol{v}}{\partial n}\right]\!\right]\Bigg|_e = \frac{\partial \boldsymbol{v}}{\partial n_T}\Bigg|_T + \frac{\partial \boldsymbol{v}}{\partial n_{T'}}\Bigg|_{T'}.$$

Let $\psi_e \in W_h^2$ be the quadratic bump function associated with the edge e such that $\psi_e(\boldsymbol{m}_{e'}) = \delta_{e,e'}$, where $\boldsymbol{m}_{e'}$ is the midpoint of e', and define the local space

$$P_2^0(T) = \operatorname{span}\{\psi_e : e \subset \partial T\}.$$

It is well known that the following useful inequalities hold for $v \in P_2^0(T)$

$$\|v\|_{0,T} \leq Ch_T\|\nabla v\|_{0,T}, \qquad \|v\|_{0,\partial T} \leq Ch_T^{1/2}\|\nabla v\|_{0,T} \tag{3.1}$$

which can be derived by the standard scaling argument.

Now we are ready to define the local Poisson problems and the corresponding error estimator.

Definition 1. *For every $T \in \mathcal{T}_h$, find $\boldsymbol{\varepsilon}_T \in (P_2^0(T))^2$ such that for all $\boldsymbol{v} \in (P_2^0(T))^2$,*

$$(\nabla\boldsymbol{\varepsilon}_T, \nabla\boldsymbol{v})_T = (\boldsymbol{f} - \nabla p_h, \boldsymbol{v})_T - \frac{1}{2}\left\langle \left[\!\!\left[\frac{\partial \boldsymbol{u}_h}{\partial n}\right]\!\!\right], \boldsymbol{v}\right\rangle_{\partial T\setminus\partial\Omega}, \tag{3.2}$$

and compute the error estimator

$$\eta_P = \left(\sum_{T\in\mathcal{T}_h} \|\nabla\boldsymbol{\varepsilon}_T\|_{0,T}^2\right)^{1/2} + \|\nabla\cdot\boldsymbol{u}_h\|_{0,\Omega}.$$

The local problem (3.2) always has a unique solution because $P_2^0(T)$ does not contain constants. Besides, it decouples into two independent Poisson problems which require solving 3×3 matrix systems.

The local efficiency of the error estimator η_P can be derived by comparing it with the residual error estimator. To show that η_P is reliable, we make the following saturation assumption which is similar to the one of Bank and Welfert [12]:

Saturation Assumption: Let $(\boldsymbol{u}_2, p_1) \in (W_h^2 \cap H_0^1(\Omega))^2 \times Q_h$ be the Taylor–Hood finite element approximation to $(\boldsymbol{u}, p)$. Then there is a constant $0 \leq \beta < 1$ independent of the mesh size h such that

$$\|\nabla(\boldsymbol{u} - \boldsymbol{u}_2)\|_{0,\Omega} + \|p - p_1\|_{0,\Omega} \leq \beta(\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega} + \|p - p_h\|_{0,\Omega}). \tag{3.3}$$

Observe that the P_2/P_1 finite element is adopted here, while the stabilized P_2/P_2 finite element is used in [12]. By the a priori error estimate (2.4) we expect that this assumption holds for sufficiently small h, at least when $\boldsymbol{u}$ and p are regular.

In the following theorem we prove that the error estimator η_P is reliable and efficient in the usual sense.

Theorem 3.1. *Let $\boldsymbol{f}_h$ be any piecewise polynomial approximation of $\boldsymbol{f}$. Then the following local lower bound holds*

$$\|\nabla\boldsymbol{\varepsilon}_T\|_{0,T} + \|\nabla\cdot\boldsymbol{u}_h\|_{0,T} \leq C(\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\omega_T} + \|p - p_h\|_{0,\omega_T} + h_T\|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,T}),$$

where ω_T is the union of T and those triangles sharing edges with T. Moreover, under the saturation assumption (3.3), we have

$$\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} \leq C\eta_P.$$

Proof. Taking $\boldsymbol{v} = \boldsymbol{\varepsilon}_T$ in (3.2) and applying the inequalities (3.1), we obtain

$$\|\nabla \boldsymbol{\varepsilon}_T\|_{0,T} \le C h_T \|\boldsymbol{f} - \nabla p_h\|_{0,T} + C h_T^{1/2} \left\| \left[\!\left[\frac{\partial \boldsymbol{u}_h}{\partial n} \right]\!\right] \right\|_{0,\partial T \setminus \partial\Omega}.$$

Since the right-hand side represents the standard residual error estimator, it can be further bounded as in the proof of Theorem 3 of [15]. Besides, we have

$$\|\nabla \cdot \boldsymbol{u}_h\|_{0,T} = \|\nabla \cdot (\boldsymbol{u}_h - \boldsymbol{u})\|_{0,T} \le 2\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,T}.$$

This proves the first result.

On the other hand, if $\boldsymbol{w} \in (W_h^2 \cap H_0^1(\Omega))^2$ vanishes at the vertices of $\mathcal{T}_h$, then the integration by parts yields

$$(\boldsymbol{f}, \boldsymbol{w})_\Omega - (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{w})_\Omega + (\nabla \cdot \boldsymbol{w}, p_h)_\Omega = \sum_{T \in \mathcal{T}_h} (\nabla \boldsymbol{\varepsilon}_T, \nabla \boldsymbol{w})_T.$$

Hence, for all $\boldsymbol{v} \in (W_h^2 \cap H_0^1(\Omega))^2$ and $q \in Q_h$, it holds that

$$\begin{aligned}
(\nabla(\boldsymbol{u}_2 - \boldsymbol{u}_h), \nabla \boldsymbol{v})_\Omega &- (\nabla \cdot \boldsymbol{v}, p_1 - p_h)_\Omega \\
&= (\boldsymbol{f}, \boldsymbol{v})_\Omega - (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v})_\Omega + (\nabla \cdot \boldsymbol{v}, p_h)_\Omega \\
&= (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{v}_I)_\Omega - (\nabla \boldsymbol{u}_h, \nabla(\boldsymbol{v} - \boldsymbol{v}_I))_\Omega + (\nabla \cdot (\boldsymbol{v} - \boldsymbol{v}_I), p_h)_\Omega \\
&= \sum_{T \in \mathcal{T}_h} (\nabla \boldsymbol{\varepsilon}_T, \nabla(\boldsymbol{v} - \boldsymbol{v}_I))_T \\
(\nabla \cdot (\boldsymbol{u}_2 - \boldsymbol{u}_h), q)_\Omega &= -(\nabla \cdot \boldsymbol{u}_h, q)_\Omega
\end{aligned}$$

since $\boldsymbol{v} - \boldsymbol{v}_I \in (W_h^2 \cap H_0^1(\Omega))^2$ vanishes at the vertices of $\mathcal{T}_h$. By stability of the Taylor–Hood element and the inequality $\|\nabla(\boldsymbol{v} - \boldsymbol{v}_I)\|_{0,\Omega} \le C\|\nabla \boldsymbol{v}\|_{0,\Omega}$, we obtain

$$\|\nabla(\boldsymbol{u}_2 - \boldsymbol{u}_h)\|_{0,\Omega} + \|p_1 - p_h\|_{0,\Omega} \le C\left\{ \left(\sum_{T \in \mathcal{T}_h} \|\nabla \boldsymbol{\varepsilon}_T\|_{0,T}^2 \right)^{1/2} + \|\nabla \cdot \boldsymbol{u}_h\|_{0,\Omega} \right\} = C\eta_P.$$

Finally, combining the saturation assumption (3.3) and the triangle inequality gives

$$\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} \le \frac{1}{1-\beta}(\|\nabla(\boldsymbol{u}_2 - \boldsymbol{u}_h)\|_{0,\Omega} + \|p_1 - p_h\|_{0,\Omega}),$$

from which the second result follows. □

Now we suppose that the triangulations $\{\mathcal{T}_h\}_{h>0}$ satisfy the Condition (α, σ) and establish the asymptotic exactness of the first term $\left(\sum_{T \in \mathcal{T}_h} \|\nabla \boldsymbol{\varepsilon}_T\|_{0,T}^2\right)^{1/2}$. Following the techniques of [19, 20], we introduce an auxiliary function defined in a similar way to $\boldsymbol{\varepsilon}_T$: for given $\boldsymbol{w} \in (H^2(\omega_T))^2$, let $\boldsymbol{q}_{\boldsymbol{w}} \in (P_2^0(T))^2$ be such that for all $\boldsymbol{v} \in (P_2^0(T))^2$,

$$(\nabla \boldsymbol{q}_{\boldsymbol{w}}, \nabla \boldsymbol{v})_T = (-\Delta \boldsymbol{w}, \boldsymbol{v})_T - \frac{1}{2}\left\langle \left[\!\left[\frac{\partial \boldsymbol{w}_I}{\partial n} \right]\!\right], \boldsymbol{v} \right\rangle_{\partial T \setminus \partial\Omega}. \tag{3.4}$$

Recall that $\boldsymbol{w}_I \in (W_h^1)^2$ denotes the standard nodal interpolant of $\boldsymbol{w}$.

The following lemma generalizes the result of [19] for uniform meshes and has the same form as Lemma 6.4 and (6.15) of [20] given for the equilibrated residual method. The proof goes in almost the same way as in [20] and is given here for the reader's convenience.

Lemma 3.2. *For every $T \in \mathcal{T}_h$, we have*

$$\|\nabla \boldsymbol{q_w}\|_{0,T} \le C h_T \|\boldsymbol{w}\|_{2,\omega_T}. \tag{3.5}$$

If $T \in \mathcal{T}_h$ has no boundary edges and all triangles of ω_T belong to $\mathcal{T}_{1,h}$, then we have for $\boldsymbol{w} \in (H^3(\omega_T))^2$

$$\|\nabla(\boldsymbol{w} - \boldsymbol{w}_I) - \nabla \boldsymbol{q_w}\|_{0,T} \le C h_T^{1+\min(\alpha,1)} \|\boldsymbol{w}\|_{3,\omega_T}. \tag{3.6}$$

Proof. Since $[\![\frac{\partial \boldsymbol{w}}{\partial n}]\!] = 0$ on $\partial T \setminus \partial\Omega$ for $\boldsymbol{w} \in (H^2(\omega_T))^2$, the equation (3.4) becomes

$$(\nabla \boldsymbol{q_w}, \nabla \boldsymbol{v})_T = (-\Delta \boldsymbol{w}, \boldsymbol{v})_T + \frac{1}{2}\left\langle \left[\!\!\left[\frac{\partial}{\partial n}(\boldsymbol{w} - \boldsymbol{w}_I) \right]\!\!\right], \boldsymbol{v} \right\rangle_{\partial T \setminus \partial\Omega}.$$

The first result (3.5) is obtained by taking $\boldsymbol{v} = \boldsymbol{q_w}$ and applying the interpolation error estimate for $\boldsymbol{w} - \boldsymbol{w}_I$ and the inequalities (3.1).

Now we turn to the second result (3.6). Let $\boldsymbol{z} \in (P_2(\omega_T))^2$. Then the integration by parts give for all $\boldsymbol{v} \in (P_2^0(T))^2$

$$\begin{aligned}
(\nabla(\boldsymbol{z} - \boldsymbol{z}_I - \boldsymbol{q_z}), \nabla \boldsymbol{v})_T &= \left\langle \frac{\partial}{\partial n_T}(\boldsymbol{z} - \boldsymbol{z}_I), \boldsymbol{v} \right\rangle_{\partial T} + \frac{1}{2}\left\langle \left[\!\!\left[\frac{\partial \boldsymbol{z}_I}{\partial n} \right]\!\!\right], \boldsymbol{v} \right\rangle_{\partial T} \\
&= \left\langle \frac{\partial}{\partial n_T}(\boldsymbol{z} - \{\!\{\boldsymbol{z}_I\}\!\}), \boldsymbol{v} \right\rangle_{\partial T} \\
&= \sum_{e \subset \partial T} \frac{2}{3}|e| \, \frac{\partial}{\partial n_T}(\boldsymbol{z} - \{\!\{\boldsymbol{z}_I\}\!\})(\boldsymbol{m}_e) \cdot \boldsymbol{v}(\boldsymbol{m}_e),
\end{aligned}$$

where $\{\!\{\boldsymbol{v}\}\!\}|_e = \frac{1}{2}(\boldsymbol{v}|_T + \boldsymbol{v}|_{T'})$ for $e = \partial T \cap \partial T'$ and $|e|$ denotes the length of e. By using the estimate (see, for example, Lemma 7.1 of [20])

$$\left|\nabla \boldsymbol{z}(\boldsymbol{m}_e) - \{\!\{\nabla \boldsymbol{z}_I\}\!\}|_e\right| \le C h_T^{1+\alpha} |\boldsymbol{z}|_{2,\infty,\omega_T},$$

it follows that

$$\begin{aligned}
|(\nabla(\boldsymbol{z} - \boldsymbol{z}_I - \boldsymbol{q_z}), \nabla \boldsymbol{v})_T| \le C h_T^{2+\alpha} |\boldsymbol{z}|_{2,\infty,\omega_T} \|\boldsymbol{v}\|_{0,\infty,T} &\le C h_T^{\alpha} |\boldsymbol{z}|_{2,\omega_T} \|\boldsymbol{v}\|_{0,T} \\
&\le C h_T^{1+\alpha} |\boldsymbol{z}|_{2,\omega_T} \|\nabla \boldsymbol{v}\|_{0,T}.
\end{aligned}$$

Taking $\boldsymbol{v} = \boldsymbol{z} - \boldsymbol{z}_I - \boldsymbol{q_z}$, we obtain

$$\|\nabla(\boldsymbol{z} - \boldsymbol{z}_I - \boldsymbol{q_z})\|_{0,T} \le C h_T^{1+\alpha} |\boldsymbol{z}|_{2,\omega_T}. \tag{3.7}$$

The results (3.5) and (3.7) lead to

$$\begin{aligned}
\|\nabla(\boldsymbol{w} - \boldsymbol{w}_I) - \nabla \boldsymbol{q_w}\|_{0,T} &\le \|\nabla(\boldsymbol{w} - \boldsymbol{z}) - \nabla(\boldsymbol{w} - \boldsymbol{z})_I - \nabla \boldsymbol{q_{w-z}}\|_{0,T} \\
&\qquad + \|\nabla(\boldsymbol{z} - \boldsymbol{z}_I - \boldsymbol{q_z})\|_{0,T} \\
&\le C h_T \|\boldsymbol{w} - \boldsymbol{z}\|_{2,\omega_T} + C h_T^{1+\alpha} |\boldsymbol{z}|_{2,\omega_T}.
\end{aligned}$$

Finally, choose $\boldsymbol{z} \in (P_2(\omega_T))^2$ satisfying $\|\boldsymbol{w}-\boldsymbol{z}\|_{2,\omega_T} \le Ch_T\|\boldsymbol{w}\|_{3,\omega_T}$ and note that $\|\boldsymbol{z}\|_{2,\omega_T} \le C\|\boldsymbol{w}\|_{3,\omega_T}$. This proves the second result (3.6). $\square$

Lemma 3.3. *Let $(\boldsymbol{u}, p)$ be the solution of (2.1). Then we have for every $T \in \mathcal{T}_h$*

$$\|\nabla(\boldsymbol{q}_{\boldsymbol{u}} - \boldsymbol{\varepsilon}_T)\|_{0,T} \le C(\|\nabla(\boldsymbol{u}_I - \boldsymbol{u}_h)\|_{0,\omega_T} + h_T\|\nabla(p-p_h)\|_{0,T}).$$

Proof. Substituting $\boldsymbol{f} = -\Delta\boldsymbol{u} + \nabla p$ in (3.2) and using the inequalities (3.1), we obtain for all $\boldsymbol{v} \in (P_2^0(T))^2$

$$\begin{aligned}(\nabla(\boldsymbol{q}_{\boldsymbol{u}} - \boldsymbol{\varepsilon}_T), \nabla\boldsymbol{v})_T &= -(\nabla(p-p_h), \boldsymbol{v})_T - \frac{1}{2}\left\langle \left[\!\!\left[\frac{\partial}{\partial n}(\boldsymbol{u}_I - \boldsymbol{u}_h) \right]\!\!\right], \boldsymbol{v} \right\rangle_{\partial T\setminus\partial\Omega} \\ &\le C(\|\nabla(\boldsymbol{u}_I - \boldsymbol{u}_h)\|_{0,\omega_T} + h_T\|\nabla(p-p_h)\|_{0,T})\|\nabla\boldsymbol{v}\|_{0,T}.\end{aligned}$$

The desired result is derived by taking $\boldsymbol{v} = \boldsymbol{q}_{\boldsymbol{u}} - \boldsymbol{\varepsilon}_T$. $\square$

With the aid of the previous two lemmas we are able to prove the following result.

Theorem 3.4. *Assume that the triangulations $\{\mathcal{T}_h\}_{h>0}$ satisfy the Condition (α, σ) and $(\boldsymbol{u}, p) \in (H^3(\Omega) \cap W^{2,\infty}(\Omega))^2 \times H^2(\Omega)$. Then we have*

$$\left(\sum_{T\in\mathcal{T}_h} \|\nabla(\boldsymbol{u} - \boldsymbol{u}_h) - \nabla\boldsymbol{\varepsilon}_T\|_{0,T}^2 \right)^{1/2} \le Ch^{1+\rho}(\|\boldsymbol{u}\|_{3,\Omega} + \|\boldsymbol{u}\|_{2,\infty,\Omega} + \|p\|_{2,\Omega})$$

with $\rho = \min(\alpha, \frac{1}{2}, \frac{\sigma}{2})$. Moreover, if $\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega} \ge Ch$ for some constant $C > 0$, then it holds that

$$\left| \frac{\eta_{P,1}}{\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega}} - 1 \right| = O(h^\rho),$$

where $\eta_{P,1} := \left(\sum_{T\in\mathcal{T}_h} \|\nabla\boldsymbol{\varepsilon}_T\|_{0,T}^2\right)^{1/2}$.

Proof. The second result follows easily from the first result, as is shown in Theorem 5.3 of [20], so we only give a proof for the first result.

The triangle inequality gives

$$\begin{aligned}\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h) - \nabla\boldsymbol{\varepsilon}_T\|_{0,T} &\le \|\nabla(\boldsymbol{u} - \boldsymbol{u}_I) - \nabla\boldsymbol{q}_{\boldsymbol{u}}\|_{0,T} \\ &\quad + \|\nabla(\boldsymbol{u}_I - \boldsymbol{u}_h)\|_{0,T} + \|\nabla(\boldsymbol{q}_{\boldsymbol{u}} - \boldsymbol{\varepsilon}_T)\|_{0,T}.\end{aligned}$$

Let $\widetilde{\mathcal{T}}_{1,h} \subset \mathcal{T}_{1,h}$ be the set of all $T \in \mathcal{T}_h$ such that $\partial T \cap \partial\Omega = \emptyset$ and all triangles of ω_T belong to $\mathcal{T}_{1,h}$. Then, by virtue of (3.6) and Lemma 3.3, we obtain for $T \in \widetilde{\mathcal{T}}_{1,h}$

$$\begin{aligned}\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h) - \nabla\boldsymbol{\varepsilon}_T\|_{0,T} &\le Ch_T^{1+\min(\alpha,1)}\|\boldsymbol{u}\|_{3,\omega_T} \\ &\quad + C(\|\nabla(\boldsymbol{u}_I - \boldsymbol{u}_h)\|_{0,\omega_T} + h_T\|\nabla(p-p_h)\|_{0,T}).\end{aligned}$$

For $T \in \mathcal{T}_h \setminus \widetilde{\mathcal{T}}_{1,h}$, the inequality (3.5) and Lemma 3.3 give

$$\begin{aligned}\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h) - \nabla\boldsymbol{\varepsilon}_T\|_{0,T} &\le Ch_T\|\boldsymbol{u}\|_{2,\omega_T} \\ &\quad + C(\|\nabla(\boldsymbol{u}_I - \boldsymbol{u}_h)\|_{0,\omega_T} + h_T\|\nabla(p-p_h)\|_{0,T}).\end{aligned}$$

Consequently,

$$\sum_{T\in\mathcal{T}_h}\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)-\nabla\boldsymbol{\varepsilon}_T\|_{0,T}^2$$
$$\leq Ch^{2+2\min(\alpha,1)}\|\boldsymbol{u}\|_{3,\Omega}^2+Ch^2\sum_{T\in\mathcal{T}_h\backslash\widetilde{\mathcal{T}}_{1,h}}\|\boldsymbol{u}\|_{2,\omega_T}^2+C|\!|\!|(\boldsymbol{u}_I-\boldsymbol{u}_h,p-p_h)|\!|\!|^2.$$

Furthermore, by the Condition (α,σ) we have

$$\sum_{T\in\mathcal{T}_h\backslash\widetilde{\mathcal{T}}_{1,h}}\|\boldsymbol{u}\|_{2,\omega_T}^2\leq\Bigg(\sum_{T\in\mathcal{T}_h\backslash\widetilde{\mathcal{T}}_{1,h}}|\omega_T|\Bigg)\|\boldsymbol{u}\|_{2,\infty,\Omega}^2\leq Ch^{\min(1,\sigma)}\|\boldsymbol{u}\|_{2,\infty,\Omega}^2,$$

and the third term is bounded by invoking the superconvergence result (2.5). The proof is completed by collecting the above results. $\square$

4. Error Estimator Based on Local Stokes Problems

In this section we analyze an error estimator based on solution of the following local Stokes problems:

Definition 2. *For every $T\in\mathcal{T}_h$, find $(\boldsymbol{\varepsilon}_T^*,e_T^*)\in(P_2^0(T))^2\times P_0(T)$ such that for all $(\boldsymbol{v},s)\in(P_2^0(T))^2\times P_0(T)$,*

$$\begin{cases}(\nabla\boldsymbol{\varepsilon}_T^*,\nabla\boldsymbol{v})_T-(\nabla\cdot\boldsymbol{v},e_T^*)_T=(\boldsymbol{f}-\nabla p_h,\boldsymbol{v})_T-\dfrac{1}{2}\Big\langle\Big[\!\!\Big[\dfrac{\partial\boldsymbol{u}_h}{\partial n}\Big]\!\!\Big],\boldsymbol{v}\Big\rangle_{\partial T\backslash\partial\Omega}\\ (\nabla\cdot\boldsymbol{\varepsilon}_T^*,s)_T=-(\nabla\cdot\boldsymbol{u}_h,s)_T,\end{cases}\tag{4.1}$$

and compute the error estimator

$$\eta_S=\Bigg\{\sum_{T\in\mathcal{T}_h}(\|\nabla\boldsymbol{\varepsilon}_T^*\|_{0,T}^2+\|e_T^*\|_{0,T}^2)\Bigg\}^{1/2}.$$

The local problem (4.1) can be viewed as a Stokes problem on T with a Neumann boundary condition and require solving 7×7 matrix systems. It is straightforward to prove the local inf-sup condition

$$\inf_{q\in P_0(T)}\sup_{\boldsymbol{v}\in(P_2^0(T))^2}\frac{(\nabla\cdot\boldsymbol{v},q)_T}{\|\nabla\boldsymbol{v}\|_{0,T}\|q\|_{0,T}}\geq m_L>0,$$

so the problem (4.1) is well-posed for every $T\in\mathcal{T}_h$.

An error estimator of this type was first proposed by Verfürth [11] for the mini element who used the quadratic bump and cubic bubble functions to solve the local Stokes problems. In [12] Bank and Welfert considered the stabilized forms of the local Stokes problems using the quadratic bump functions only for both the velocity and pressure errors. Our error estimator uses the P_2^0/P_0 element and is very similar to the one of Kay and Silvester [17] (proposed for the stabilized P_1/P_0 finite element method) which uses the P_2^0/P_1 element.

The following theorem shows that η_S is locally equivalent to η_P. This, in particular, implies that Theorem 3.1 is valid for η_S as well as for η_P.

Theorem 4.1. *Let $\boldsymbol{\varepsilon}_T$ be defined by (3.2) and let $(\boldsymbol{\varepsilon}_T^*, e_T^*)$ be defined by (4.1). Then we have for every $T \in \mathcal{T}_h$*

$$C_1(\|\nabla \boldsymbol{\varepsilon}_T\|_{0,T} + \|\nabla \cdot \boldsymbol{u}_h\|_{0,T}) \le \|\nabla \boldsymbol{\varepsilon}_T^*\|_{0,T} + \|e_T^*\|_{0,T} \le C_2(\|\nabla \boldsymbol{\varepsilon}_T\|_{0,T} + \|\nabla \cdot \boldsymbol{u}_h\|_{0,T}).$$

Proof. By (3.2) and (4.1), we have for $(\boldsymbol{v}, s) \in (P_2^0(T))^2 \times P_0(T)$

$$\begin{cases} (\nabla \boldsymbol{\varepsilon}_T^*, \nabla \boldsymbol{v})_T - (\nabla \cdot \boldsymbol{v}, e_T^*)_T = (\nabla \boldsymbol{\varepsilon}_T, \nabla \boldsymbol{v})_T \\ (\nabla \cdot \boldsymbol{\varepsilon}_T^*, s)_T = -(\nabla \cdot \boldsymbol{u}_h, s)_T. \end{cases}$$

Thus the right inequality is a direct consequence of the well-posedness of the local problem (4.1). The left inequality follows easily by taking $\boldsymbol{v} = \boldsymbol{\varepsilon}_T$, $s = \nabla \cdot \boldsymbol{u}_h$ and then applying the Cauchy–Schwarz inequality. □

Finally, we prove the following analogue of Theorem 3.4 which states that the velocity component of the error estimator η_S is asymptotically exact.

Theorem 4.2. *Under the assumptions of Theorem 3.4, we have*

$$\Big(\sum_{T\in\mathcal{T}_h} \|\nabla(\boldsymbol{u} - \boldsymbol{u}_h) - \nabla \boldsymbol{\varepsilon}_T^*\|_{0,T}^2\Big)^{1/2} + \Big(\sum_{T\in\mathcal{T}_h} \|e_T^*\|_{0,T}^2\Big)^{1/2} \le Ch^{1+\rho}(\|\boldsymbol{u}\|_{3,\Omega} + \|\boldsymbol{u}\|_{2,\infty,\Omega} + \|p\|_{2,\Omega}) \quad (4.2)$$

with $\rho = \min(\alpha, \frac{1}{2}, \frac{\sigma}{2})$. Moreover, if $\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega} \ge Ch$ for some constant $C > 0$, then it holds that

$$\left| \frac{\eta_{S,1}}{\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega}} - 1 \right| = O(h^\rho),$$

where $\eta_{S,1} := \big(\sum_{T\in\mathcal{T}_h} \|\nabla \boldsymbol{\varepsilon}_T^*\|_{0,T}^2\big)^{1/2}$.

Proof. Let $\boldsymbol{\varepsilon}_T$ be defined by (3.2). By (3.2), (4.1) and the incompressibility condition $\nabla\cdot\boldsymbol{u} = 0$, we have for $(\boldsymbol{v}, s) \in (P_2^0(T))^2 \times P_0(T)$

$$\begin{cases} (\nabla(\boldsymbol{\varepsilon}_T^* - \boldsymbol{\varepsilon}_T), \nabla \boldsymbol{v})_T - (\nabla \cdot \boldsymbol{v}, e_T^*)_T = 0 \\ (\nabla \cdot (\boldsymbol{\varepsilon}_T^* - \boldsymbol{\varepsilon}_T), s)_T = (\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_h - \boldsymbol{\varepsilon}_T), s)_T. \end{cases}$$

Then it follows by the well-posedness of the local problem (4.1) that

$$\|\nabla(\boldsymbol{\varepsilon}_T^* - \boldsymbol{\varepsilon}_T)\|_{0,T} + \|e_T^*\|_{0,T} \le C\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h - \boldsymbol{\varepsilon}_T)\|_{0,T}$$

and thus

$$\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h) - \nabla \boldsymbol{\varepsilon}_T^*\|_{0,T} + \|e_T^*\|_{0,T} \le C\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h) - \nabla \boldsymbol{\varepsilon}_T\|_{0,T}.$$

This proves the first result (4.2) by Theorem 3.4. The second result follows from (4.2), as stated in the proof of Theorem 3.4. □

Remark 2. *The superconvergence result (2.5) and the estimate (4.2) indicates that the pressure error* $\|p - p_h\|_{0,\Omega}$ *and the pressure component* $\left(\sum_{T\in\mathcal{T}_h} \|e_T^*\|_{0,T}^2\right)^{1/2}$ *of* η_S *are both of the order* $O(h^{1+\rho})$ *under the assumptions of Theorem 3.4, so they becomes negligible compared with their velocity counterparts as* $h \to 0$*. This leads to*

$$\left|\frac{\eta_S}{\left(\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega}^2 + \|p - p_h\|_{0,\Omega}^2\right)^{1/2}} - 1\right| = O(h^\rho).$$

In other words, η_S *is asymptotically exact with respect to the total error.*

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