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TRANSFORMATION OF DIMENSIONLESS HEAT DIFFUSION EQUATION FOR THE SOLUTION OF DYNAMIC DOMAIN IN PHASE CHANGE PROBLEMS

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ABSTRACT. In the present work transformation of dimensionless heat diffusion equation for the solution of moving boundary problems have been formulated. The formulation is based on 1-D, 2-D and 3-D, unsteady heat diffusion equations. These equations are first turned into dimensionless form by using dimensionless quantities and their transformation was formulated in liquid and solid phases. The salient feature of this work is that during the transformation of dimensionless heat diffusion equation there arises a convective term $\tilde{v}$ which is responsible for the motion of interface in liquid as well as solid phase. In the transformed heat equation, a correction factor β also arises naturally which gives the correct transformed flux at interface.

1. INTRODUCTION

Transient heat transfer problems, involving melting or solidification, generally referred to as phase-change or moving body problems are important in many engineering applications. The solution of such problem is difficult because the interface between the two phases is moving as latent heat is absorbed or released at the interface [1]. For the numerical solution of such problems dimensionless form of partial differential equation is required. The problems involving moving boundaries are very complicated at the interface, so transformation of dimensionless partial differential equation is a helpful technique for their solution in liquid and solid domain [4]. During the process of melting or solidification, interface moves in between liquid and solid part of the domain. A transformation technique must therefore be used to insure that the final projection from the old to new domain is exact. The basis of above mentioned transformation are the classical heat equations $T_1(x, y, z, t)$ and $T_2(x, y, z, t)$ [1].

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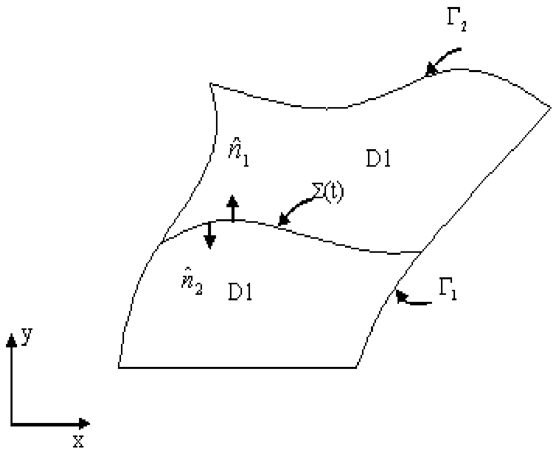

FIGURE 1. Domain definition for a two dimensional moving boundary problem.

$$\rho C_P \frac{\partial T_1}{\partial t} = K_1 \nabla^2 T_1 \text{ in } D_1 \tag{1}$$

$$\rho C_P \frac{\partial T_2}{\partial t} = K_2 \nabla^2 T_2 \text{ in } D_2 \tag{2}$$

$$\rho L V_n = K_2 \nabla T_2 \cdot \hat{n}_2 - K_1 \nabla T_1 \cdot \hat{n}_1 \text{ on } \sum(t). \tag{3}$$

Here ρ is the density, C_P is the specific heat, and K is the thermal conductivity, L is the latent heat of fusion and v_n is the interface velocity in the direction $\hat{n}_1$ normal to the surface $\sum(t)$. The subscripts 1 and 2 refer to the liquid and solid phases as from Figure 1, respectively. The initial conditions on the temperature and the interface position are given by

$$T_1(x, y, z, t = 0) \tag{4}$$

$$T_2(x, y, z, t = 0) \tag{5}$$

$$\sum(t = 0) = \overset{o}{\sum}. \tag{6}$$

And the boundary conditions are assumed to be of the form:

$$a_1 \nabla T_1 \cdot n_1 = c_1 \text{ on } \Gamma_1 \tag{7}$$

$$a_2 \nabla T_2 \cdot n_2 = c_2 \text{ on } \Gamma_2 \tag{8}$$

$$T_1 = T_2 = T_{m,s}. \tag{9}$$

Here a_i, b_i, and c_i are constants and $T_{m,s}$ is the melting or solidification temperature of the material. Γ_1 and Γ_2 are the liquid and solid boundaries of the domain.

2. DIMENSIONLESS FORM OF THE HEAT EQUATIONS

The dimensionless form of the 1-D, 2-D, and 3-D heat diffusion equations by using dimensionless quantities is given as under:

2.1. **Dimensional form of 1-D Heat Equation.** The dimensionless form non dimensional heat diffusion equation is as follows:

$$\frac{\partial \Theta_1}{\partial \tau} = \frac{\partial^2 \Theta_1}{\partial \xi^2},\ 0 < \xi < \eta \tag{10}$$

$$\frac{\partial \Theta_2}{\partial \tau} = \frac{\partial^2 \Theta_2}{\partial \xi^2},\ \eta < \xi < 1 \tag{11}$$

$$\gamma \frac{d\eta}{d\tau} = -\frac{\partial \Theta_1}{\partial \xi} + K \frac{\partial \Theta_2}{\partial \xi}\ \xi = \eta. \tag{12}$$

2.2. **Dimensional form of 2-D Heat Equation.** The dimensionless form two dimensional heat diffusion equation is as follows:

$$\frac{\partial \Theta_1}{\partial \tau} = \frac{\partial^2 \Theta_1}{\partial \xi_1^2} + \frac{\partial^2 \Theta_1}{\partial \xi_2^2},\ 0 < \xi < \eta, 0 < \xi_2 < \eta \tag{13}$$

$$\frac{\partial \Theta_2}{\partial \tau} = K\left(\frac{\partial^2 \Theta_2}{\partial \xi_1^2} + \frac{\partial^2 \Theta_2}{\partial \xi_2^2}\right),\ 0 < \xi < \eta, 0 < \xi_2 < \eta \tag{14}$$

$$\gamma \frac{d\eta}{d\tau} = -\left(\frac{\partial \Theta_1}{\partial \xi_1} + \frac{\partial \Theta_1}{\partial \xi_2}\right) + K\left(\frac{\partial \Theta_2}{\partial \xi_1} + \frac{\partial \Theta_2}{\partial \xi_2}\right),\ \xi_1 = \eta, \xi_2 = \eta. \tag{15}$$

2.3. **Dimensional form of 3-D Heat Equation.** The dimensionless form three dimensional heat diffusion equation is as follows:

$$\frac{\partial \Theta_1}{\partial \tau} = \frac{\partial^2 \Theta_1}{\partial \xi_1^2} + \frac{\partial^2 \Theta_1}{\partial \xi_2^2} + \frac{\partial^2 \Theta_1}{\partial \xi_3^2},\ 0 < \xi_1 < \eta, 0 < \xi_2 < \eta, 0 < \xi_3 < \eta \tag{16}$$

$$\frac{\partial \Theta_2}{\partial \tau} = K\left(\frac{\partial^2 \Theta_2}{\partial \xi_1^2} + \frac{\partial^2 \Theta_2}{\partial \xi_2^2} + \frac{\partial^2 \Theta_2}{\partial \xi_3^2}\right),\ 0 < \xi_1 < \eta, 0 < \xi_2 < \eta, 0 < \xi_3 < \eta \tag{17}$$

$$\gamma \frac{d\eta}{d\tau} = -\left(\frac{\partial \Theta_1}{\partial \xi_1} + \frac{\partial \Theta_1}{\partial \xi_2} + \frac{\partial \Theta_1}{\partial \xi_3}\right) + K\left(\frac{\partial \Theta_2}{\partial \xi_1} + \frac{\partial \Theta_2}{\partial \xi_2} + \frac{\partial \Theta_2}{\partial \xi_3}\right),\ \xi_1 = \eta, \xi_2 = \eta, \xi_3 = \eta. \tag{18}$$

Dimensionless quantities are:

$$\xi = \frac{x}{l},\ \tau = \frac{\alpha_1 t}{l^2},\ \Theta_1 = \left(\frac{T_1 - T_b}{\triangle T}\right)$$

$$\Theta_{m,s} = \left(\frac{T_{m,s} - T_b}{\triangle T}\right),\ \gamma = \frac{L}{C_P \triangle T}$$

$$\eta = \frac{s}{l},\ \alpha = \frac{k}{\rho C_P},\ \triangle T = T_a - T_b,\ \xi_1 = \frac{x_1}{l_1},\ \xi_2 = \frac{x_2}{l_2},\ \xi_3 = \frac{x_3}{l_3}$$

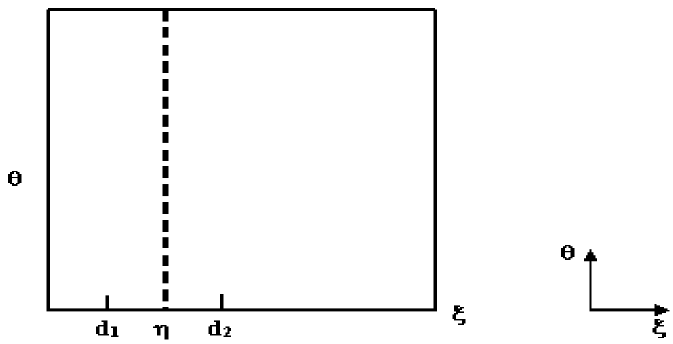

FIGURE 2. The dynamic regions $d_1 < \xi < \eta(\tau)$ and $\eta(\tau) < \xi < d_2$ are transformed to fixed domains through an interface-local transformation technique.

$$\Theta_2 = \left(\frac{T_2 - T_b}{\triangle T}\right).$$

3. TRANSFORMATION OF DIMENSIONLESS HEAT DIFFUSION EQUATIONS

The transformation of dimensionless heat diffusion equation is given as under:

3.1. **Transformation of 1-D dimensionless Heat Diffusion Equation.** The dimensionless form of the heat diffusion equation for the liquid part of the domain is as under:

$$\frac{\partial \Theta_1}{\partial \tau} = \frac{\partial^2 \Theta_1}{\partial \xi^2},\ 0 < \xi < \eta. \tag{19}$$

The dimensionless heat equations are defined on dynamic domains described by the motion of the interface boundary $\sum(t)$. In time dependent moving mesh technique, at every time step the heat equation in each phase (4-8) are first solved on a fixed domain corresponding to the old position of the phase interface. The new interface position is then calculated on the basis of (4c-8c), the geometry and domain of the problem are updated. A transformation technique must therefore be used to insure that the final projection from the old to new domain is exact [4]. We now present the particular interface local transformation technique that we have employed; as the mapping procedure is identical in both phases. We consider only the liquid phase and drop the subscript 1, [2]

$$\frac{\partial \Theta}{\partial \tau} = \frac{\partial^2 \Theta_1}{\partial \xi^2},\ 0 < \xi < \eta(\tau). \tag{20}$$

Let ξ be the coordinate on the time dependent domain and $\tilde{\xi}$ be the coordinate on the associated fixed (old) domain. Our transformation is then given by:

$$\tilde{\xi} = \xi,\ 0 < \xi < d_1 \tag{21}$$

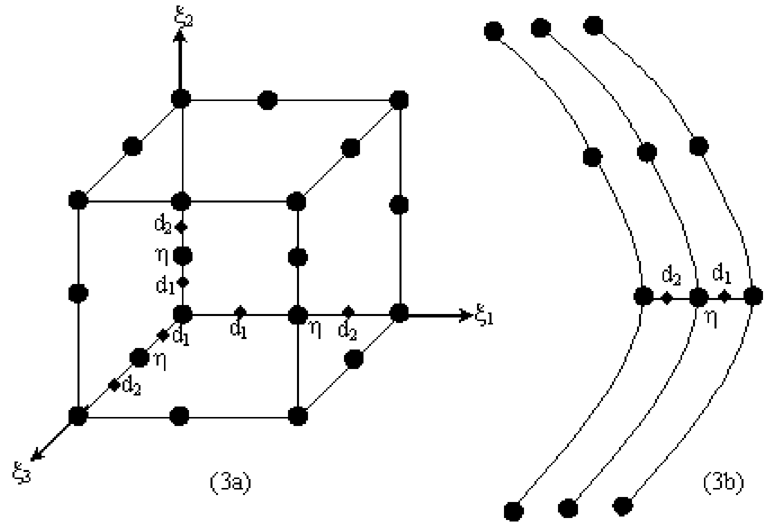

FIGURE 3. (3a) The dynamic regions $d_1 < \xi_1, \xi_2, \xi_3 < \eta(\tau)$ and $\eta(\tau) < \xi_1, \xi_2, \xi_3 < d_2$ are transformed to fixed domains through an interface-local transformation technique in 3D. (3b) Interpretation of the moving interface, η is the position of the interface.

$$\tilde{\xi} = \left(\frac{\eta(\tau_o) - d_1}{\eta(\tau) - d_1}\right)(\xi - d_1) + d_1, \ d_1 < \xi < \eta(\tau) \tag{22}$$

which maps $\xi \in [0, \eta(\tau)]$ to $\tilde{\xi} \in [0, \eta(\tau_o)]$, as shown in Figure 2. The transformation involves only that part of the dynamic domain $d_1 < \xi < \eta(\tau)$, with d_1 chosen near the interface. Where $\eta(\tau_o)$ and $\eta(\tau)$ corresponds to the interface position at the old time step and at new time step respectively [2,7,8].

If we see that the left hand side of the equation (20) can be written as:

$$\frac{\partial \Theta}{\partial \tau} = \frac{\partial \tilde{\Theta}}{\partial \tilde{\xi}} \frac{d\tilde{\xi}}{d\tau} + \frac{\partial \tilde{\Theta}}{\partial \tau} \tag{23}$$

where $\frac{\partial \tilde{\Theta}}{\partial \tau}$ is the temporal term in dynamic domain, and by using (22) we have

$$\frac{d\tilde{\xi}}{d\tau} = -\frac{(\eta(\tau_o) - d_1)}{(\eta(\tau) - d_1)^2} \frac{d\tau(\eta)}{d\tau}(\xi - d_1). \tag{24}$$

Thus equation (23) can be written as:

$$\frac{\partial \Theta}{\partial \tau} = -\frac{(\eta(\tau_o) - d_1)}{(\eta(\tau) - d_1)^2} \frac{d\tau(\eta)}{d\tau}(\xi - d_1)\frac{\partial \tilde{\Theta}}{\partial \tilde{\xi}} + \frac{\partial \tilde{\Theta}}{\partial \tau}. \tag{25}$$

We can also write

$$\frac{\partial^2\Theta}{\partial\xi^2} = \left(\frac{\eta(\tau_o)-d_1}{\eta(\tau)-d_1}\right)^2 \frac{\partial^2\tilde{\Theta}}{\partial\tilde{\xi}^2}. \tag{26}$$

By substituting (25) and (26) in (20), we have the equation of the form

$$\frac{\partial\tilde{\Theta}}{\partial\tau} - \frac{(\eta(\tau_o)-d_1)}{(\eta(\tau)-d_1)^2}\frac{d\tau(\eta)}{d\tau}(\xi-d_1)\frac{\partial\tilde{\Theta}}{\partial\tilde{\xi}} = \left(\frac{\eta(\tau_o)-d_1}{\eta(\tau)-d_1}\right)^2 \frac{\partial^2\tilde{\Theta}}{\partial\tilde{\xi}^2}. \tag{27}$$

The equation (27) can be arranged as follows:

$$\frac{1}{\beta}\frac{\partial\tilde{\Theta}}{\partial\tau} + \tilde{v}\frac{\partial\tilde{\Theta}}{\partial\tilde{\xi}} = \beta\frac{\partial^2\tilde{\Theta}}{\partial\tilde{\xi}^2} \tag{28}$$

$$\tilde{v} = -\frac{1}{\beta}\frac{d\tau(\eta)}{d\tau}$$

$$\beta = \frac{\xi-d_1}{\tau(\eta)-d_1}.$$

The motivation behind the transformation is identity $\tilde{\Theta}(\tilde{\xi},\tau)=\Theta(\xi,\tau)$ that is, any solution of convection-diffusion $0<\tilde{\xi}<\eta(\tau_o)$ is identical to the solution of the heat diffusion on the new domain $0<\tilde{\xi}<\eta(\tau)$. This implies that the solution on the the old fixed domain can be projected directly on the new domain .Where the additional convective term $\tilde{v}$ reflects that the domain is fixed near the phase interface. The correction factor β in the transformed heat equation are distributed so that the equivalent variational statement naturally gives the correct transformed flux at the interface.

3.2. **Transformation of 2-D dimensionless Heat Diffusion Equation.** The 2-D, dimensionless form of the heat diffusion equation for the liquid part of the domain is as under:

$$\frac{\partial\Theta}{\partial\tau} = \frac{\partial^2\Theta_1}{\partial\xi_1^2} + \frac{\partial^2\Theta_1}{\partial\xi_2^2},\ 0<\xi_1<\eta(\tau), 0<\xi_2<\eta(\tau). \tag{29}$$

Let ξ_1,and ξ_2 be the coordinate on the time dependent domain $\tilde{\xi}_1$ and $\tilde{\xi}_2$ be the coordinates on the associated fixed old domain, the transformation is thus:

$$\tilde{\xi}_1 = \xi_1,\ 0<\xi_1<d_1 \tag{30}$$

$$\tilde{\xi}_2 = \xi_2,\ 0<\xi_2<d_1 \tag{31}$$

and

$$\tilde{\xi}_1 = \left(\frac{\eta(\tau_o)-d_1}{\eta(\tau)-d_1}\right)(\xi_1-d_1)+d_1,\ d_1<\xi_1<\eta(\tau) \tag{32}$$

$$\tilde{\xi}_2 = \left(\frac{\eta(\tau_o)-d_1}{\eta(\tau)-d_1}\right)(\xi_2-d_1)+d_1,\ d_1<\xi_2<\eta(\tau) \tag{33}$$

and

$$\xi_1\in[0,\eta(\tau)],\xi_2\in[0,\eta(\tau)]$$

to

$$\tilde{\xi}_1 \in [0, \eta(\tau_o)]\,,\; \tilde{\xi}_2 \in [0, \eta(\tau_o)]\,.$$

The left hand side of the equation (29) can be written as

$$\frac{\partial \Theta}{\partial \tau} = \frac{\partial \tilde{\Theta}}{\partial \tilde{\xi}_1}\frac{d\tilde{\xi}_1}{d\tau} + \frac{\partial \tilde{\Theta}}{\partial \tilde{\xi}_2}\frac{d\tilde{\xi}_2}{d\tau} + \frac{\partial \tilde{\Theta}}{\partial \tau}. \tag{34}$$

By using (32) and (33) the equation (34) can be written as

$$\frac{\partial \Theta}{\partial \tau} = -\frac{(\eta(\tau_o) - d_1)}{(\eta(\tau) - d_1)^2}\frac{d\tau(\eta)}{d\tau}(\xi_1 - d_1)\frac{\partial \tilde{\Theta}}{\partial \tilde{\xi}_1} - \frac{(\eta(\tau_o) - d_1)}{(\eta(\tau) - d_1)^2}\frac{d\tau(\eta)}{d\tau}(\xi_2 - d_1)\frac{\partial \tilde{\Theta}}{\partial \tilde{\xi}_2} + \frac{\partial \tilde{\Theta}}{\partial \tau}. \tag{35}$$

And the terms on the right hand side of the equation (29) can be written as

$$\frac{\partial^2 \Theta}{\partial \xi_1^2} = \left(\frac{\eta(\tau_o) - d_1}{\eta(\tau) - d_1}\right)^2 \frac{\partial^2 \tilde{\Theta}}{\partial \tilde{\xi}_1^2} \tag{36}$$

$$\frac{\partial^2 \Theta}{\partial \xi_2^2} = \left(\frac{\eta(\tau_o) - d_1}{\eta(\tau) - d_1}\right)^2 \frac{\partial^2 \tilde{\Theta}}{\partial \tilde{\xi}_2^2}. \tag{37}$$

By using equation (35), (36), and (37) the equation (29) is of the form

$$\frac{\partial \tilde{\Theta}}{\partial \tau} - \frac{(\eta(\tau_o) - d_1)}{(\eta(\tau) - d_1)^2}\frac{d\tau(\eta)}{d\tau}(\xi_1 - d_1)\frac{\partial \tilde{\Theta}}{\partial \tilde{\xi}_1} - \frac{(\eta(\tau_o) - d_1)}{(\eta(\tau) - d_1)^2}\frac{d\tau(\eta)}{d\tau}(\xi_2 - d_1)\frac{\partial \tilde{\Theta}}{\partial \tilde{\xi}_2}$$

$$= \left(\frac{\eta(\tau_o) - d_1}{\eta(\tau) - d_1}\right)^2 \frac{\partial^2 \tilde{\Theta}}{\partial \tilde{\xi}_1^2} + \frac{\partial^2 \Theta}{\partial \xi_2^2}\left(\frac{\eta(\tau_o) - d_1}{\eta(\tau) - d_1}\right)^2 \frac{\partial^2 \tilde{\Theta}}{\partial \tilde{\xi}_2^2}. \tag{38}$$

The above equation (38) can be arranged as follows

$$\frac{1}{\beta}\frac{\partial \tilde{\Theta}}{\partial \tau} + \tilde{v}_1 \frac{\partial \tilde{\Theta}}{\partial \tilde{\xi}_{\,1}} + \tilde{v}_2 \frac{\partial \tilde{\Theta}}{\partial \tilde{\xi}_{\,2}} = \beta\left(\frac{\partial^2 \tilde{\Theta}}{\partial \tilde{\xi}_1^2} + \frac{\partial^2 \tilde{\Theta}}{\partial \tilde{\xi}_2^2}\right). \tag{39}$$

The important terms in equation (39), have the values.

$$\tilde{v}_1 = -\frac{1}{\beta}\frac{d\tau(\eta)'}{d\tau}\;\; \tilde{v}_2 = -\frac{1}{\beta}\frac{d\tau(\eta)}{d\tau}$$

$$\beta = \frac{\xi_1 - d_1}{\tau(\eta) - d_1},\; \beta = \frac{\xi_2 - d_1}{\tau(\eta) - d_1}$$

where $\tilde{v}_1$ and $\tilde{v}_2$ are convective terms in two dimensions and are responsible for the movement of interface in liquid phase,and is the same for the solid phase, and the correction factor β in the transformed heat equation are distributed so that the equivalent variational statement naturally gives the correct transformed flux at the interface in two dimensions problems.

3.3. **Transformation of 3-D dimensionless Heat Diffusion Equation.** The 3-D, dimensionless form of the heat diffusion equation for the liquid part of the domain is as under:

$$\frac{\partial \Theta}{\partial \tau} = \frac{\partial^2 \Theta_1}{\partial \xi_1^2} + \frac{\partial^2 \Theta_1}{\partial \xi_2^2} + \frac{\partial^2 \Theta_1}{\partial \xi_3^2},\ 0 < \xi_1 < \eta(\tau), 0 < \xi_2 < \eta(\tau). \tag{40}$$

Let ξ_1, ξ_2, and ξ_3 be the coordinate on the time dependent domain $\tilde{\xi_1}$, $\tilde{\xi_2}$, and $\tilde{\xi_3}$ be the coordinates on the associated fixed old domain as from Figure 3, the transformation is thus:

$$\tilde{\xi_1} = \xi_1,\ 0 < \xi_1 < d_1 \tag{41}$$

$$\tilde{\xi_2} = \xi_2,\ 0 < \xi_2 < d_1 \tag{42}$$

$$\tilde{\xi_3} = \xi_3,\ 0 < \xi_3 < d_1 \tag{43}$$

$$\tilde{\xi_1} = \left(\frac{\eta(\tau_o) - d_1}{\eta(\tau) - d_1}\right)(\xi_1 - d_1) + d_1,\ d_1 < \xi_1 < \eta(\tau) \tag{44}$$

$$\tilde{\xi_2} = \left(\frac{\eta(\tau_o) - d_1}{\eta(\tau) - d_1}\right)(\xi_2 - d_1) + d_1,\ d_1 < \xi_2 < \eta(\tau) \tag{45}$$

$$\tilde{\xi_3} = \left(\frac{\eta(\tau_o) - d_1}{\eta(\tau) - d_1}\right)(\xi_3 - d_1) + d_1,\ d_1 < \xi_3 < \eta(\tau) \tag{46}$$

where

$$\xi_1 \in [0, \eta(\tau)], \xi_2 \in [0, \eta(\tau)], \xi_3 \in [0, \eta(\tau)]$$

to

$$\tilde{\xi_1} \in [0, \eta(\tau_o)], \tilde{\xi_2} \in [0, \eta(\tau_o)], \tilde{\xi_3} \in [0, \eta(\tau_o)].$$

The left hand side of the equation (40) can be written as

$$\frac{\partial \Theta}{\partial \tau} = \frac{\partial \tilde{\Theta}}{\partial \tilde{\xi_1}} \frac{d\tilde{\xi_1}}{d\tau} + \frac{\partial \tilde{\Theta}}{\partial \tilde{\xi_2}} \frac{d\tilde{\xi_2}}{d\tau} + \frac{\partial \tilde{\Theta}}{\partial \tilde{\xi_3}} \frac{d\tilde{\xi_3}}{d\tau} + \frac{\partial \tilde{\Theta}}{\partial \tau}. \tag{47}$$

By using equations (41-46) the above equation (47) can be written as

$$\begin{aligned}\frac{\partial \Theta}{\partial \tau} = &-\frac{(\eta(\tau_o) - d_1)}{(\eta(\tau) - d_1)^2} \frac{d\tau(\eta)}{d\tau}(\xi_1 - d_1)\frac{\partial \tilde{\Theta}}{\partial \tilde{\xi_1}} - \frac{(\eta(\tau_o) - d_1)}{(\eta(\tau) - d_1)^2} \frac{d\tau(\eta)}{d\tau}(\xi_2 - d_1)\frac{\partial \tilde{\Theta}}{\partial \tilde{\xi_2}} \\ &-\frac{(\eta(\tau_o) - d_1)}{(\eta(\tau) - d_1)^2} \frac{d\tau(\eta)}{d\tau}(\xi_3 - d_1)\frac{\partial \tilde{\Theta}}{\partial \tilde{\xi_3}} + \frac{\partial \tilde{\Theta}}{\partial \tau}.\end{aligned} \tag{48}$$

The terms right hand side of the equation (40), can be written as

$$\frac{\partial^2 \Theta}{\partial \xi_1^2} = \left(\frac{\eta(\tau_o) - d_1}{\eta(\tau) - d_1}\right)^2 \frac{\partial^2 \tilde{\Theta}}{\partial \tilde{\xi}_1^2} \tag{49}$$

$$\frac{\partial^2 \Theta}{\partial \xi_2^2} = \left(\frac{\eta(\tau_o) - d_1}{\eta(\tau) - d_1}\right)^2 \frac{\partial^2 \tilde{\Theta}}{\partial \tilde{\xi}_2^2} \tag{50}$$

$$\frac{\partial^2 \Theta}{\partial \xi_3^2} = \left(\frac{\eta(\tau_o) - d_1}{\eta(\tau) - d_1}\right)^2 \frac{\partial^2 \tilde{\Theta}}{\partial \tilde{\xi}_3^2}. \tag{51}$$

By using (48-51), we have the equation of the form

$$\frac{\partial\tilde{\Theta}}{\partial\tau}-\frac{(\eta(\tau_o)-d_1)}{(\eta(\tau)-d_1)^2}\frac{d\tau(\eta)}{d\tau}(\xi_1-d_1)\frac{\partial\tilde{\Theta}}{\partial\tilde{\xi_1}}-\frac{(\eta(\tau_o)-d_1)}{(\eta(\tau)-d_1)^2}\frac{d\tau(\eta)}{d\tau}(\xi_2-d_1)$$

$$\frac{\partial\tilde{\Theta}}{\partial\tilde{\xi}_2}-\frac{(\eta(\tau_o)-d_1)}{(\eta(\tau)-d_1)^2}\frac{d\tau(\eta)}{d\tau}(\xi_3-d_1)\frac{\partial\tilde{\Theta}}{\partial\tilde{\xi}_3}=\left(\frac{\eta(\tau_o)-d_1}{\eta(\tau)-d_1}\right)^2$$

$$\frac{\partial^2\tilde{\Theta}}{\partial\tilde{\xi}_1^2}+\frac{\partial^2\Theta}{\partial\xi_2^2}\left(\frac{\eta(\tau_o)-d_1}{\eta(\tau)-d_1}\right)^2\frac{\partial^2\tilde{\Theta}}{\partial\tilde{\xi}_2^2}+\frac{\partial^2\Theta}{\partial\xi_3^2}\left(\frac{\eta(\tau_o)-d_1}{\eta(\tau)-d_1}\right)^2\frac{\partial^2\tilde{\Theta}}{\partial\tilde{\xi}_3^2}. \tag{52}$$

The above equation (52) can be arranged as follows

$$\frac{1}{\beta}\frac{\partial\tilde{\Theta}}{\partial\tau}+\tilde{v}_1\frac{\partial\tilde{\Theta}}{\partial\tilde{\xi}_{\ 1}}+\tilde{v}_2\frac{\partial\tilde{\Theta}}{\partial\tilde{\xi}_{\ 2}}+\tilde{v}_3\frac{\partial\tilde{\Theta}}{\partial\tilde{\xi}_{\ 3}}=\beta\left(\frac{\partial^2\tilde{\Theta}}{\partial\tilde{\xi}_1^2}+\frac{\partial^2\tilde{\Theta}}{\partial\tilde{\xi}_2^2}+\frac{\partial^2\tilde{\Theta}}{\partial\tilde{\xi}_3^2}\right). \tag{53}$$

The important terms in equation (53), have the values

$$\tilde{v}_1=-\frac{1}{\beta}\frac{d\tau(\eta)'}{d\tau},\ \tilde{v}_2=-\frac{1}{\beta}\frac{d\tau(\eta)}{d\tau},\tilde{v}_3=-\frac{1}{\beta}\frac{d\tau(\eta)}{d\tau}$$

$$\beta=\frac{\xi_1-d_1}{\tau(\eta)-d_1},\ \beta=\frac{\xi_2-d_1}{\tau(\eta)-d_1},\beta=\frac{\xi_3-d_1}{\tau(\eta)-d_1}.$$

Conclusion:

The most important feature of this formulation is that convective term arises during transformation which is responsible for the motion of interface in the phase change problem with liquid as well as solid phase. The terms $\tilde{v}_1$, $\tilde{v}_2$, and $\tilde{v}_3$, play an significant role in heat transfer and hence affect the progress of solidification as well melting front. This approach provides the fundamental governing equations (28), (39), and (53) in the form of transformation of dimensionless heat diffusion equation and can be used to analyze the complex geometries by using different numerical techniques. The correction factor β which is also arises naturally which gives the correct transformed flux at the interface. Thus this formulation is a golden standard for numerical analyst to analyze the moving boundary problems.

Nomonclature:

T_1, T_2	$=$	Temperature in the liquid and solid phase
$T_{m,s}$	$=$	Melting or solidification temperature
ρ	$=$	Fluid density
Γ_1, Γ_2	$=$	Liquid and solid boundaries
∇T	$=$	Temperature difference
C_P	$=$	Specific heat
v_n	$=$	Interface velocity
γ	$=$	Stefan number
L	$=$	Latent heat, amount of heat release or absorbed by interface
Θ_1, Θ_2	$=$	Dimensionless liquid and solid domain temperature
k	$=$	Thermal Conductivity

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