

MISCLASSIFICATION IN SIZE-BIASED MODIFIED POWER SERIES DISTRIBUTION AND ITS APPLICATIONS

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ABSTRACT. A misclassified size-biased modified power series distribution (MSBMPSD) where some of the observations corresponding to $x = c + 1$ are misclassified as $x = c$ with probability α , is defined. We obtain its recurrence relations among the raw moments, the central moments and the factorial moments. Discussion of the effect of the misclassification on the variance is considered. To illustrate the situation under consideration some of its particular cases like the size-biased generalized negative binomial (SBGNB), the size-biased generalized Poisson (SBGP) and size-biased Borel distributions are included. Finally, an example is presented for the size-biased generalized Poisson distribution to illustrate the results.

1. INTRODUCTION

In certain experimental investigations involving discrete distributions external factors may induce a measurement error in the form of misclassification. For instance, a situation may arise where certain values are erroneously reported; such a situation termed as modified or misclassified has been studied by Cohen ([1],[2],[3] for the Poisson and the binomial random variables, Jani and Shah [4] for modified power series distribution (MPSD) where some of the value of one are sometimes reported as zero, and recently by Patel and Patel ([5], [6]) incase of generalized power series distribution (GPSD) and MPSD for a more general situation where sometimes the value ($c + 1$) is reported erroneously as c .

Cohen [2] altered data from Bortkiewicz's [7] classical example on deaths from the kick of a horse in the Prussian Army, to illustrate the practical application of his results. He assumed that twenty of 200 given records which should have shown one death were in error by reporting no deaths. The same example was considered by Williford and Bingham [8].

In this paper we are concerned with the situation where sometimes the value ($c + 1$) is reported erroneously as c in relation to size-biased MPSD. As we know, weighted distributions arise when the observations generated from a stochastic process are not given equal chance of being recorded; instead, they are recorded according to some weight function.

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When the weight function depends on the lengths of the units of interest, the resulting distribution is called length biased. More generally, when the sampling mechanism selects units with probability proportional to some measure of the unit size, the resulting distribution is called size-biased. Such distributions arise in life length studies and were studied by various authors (see Blumenthal [9], Scheaffer [10], Gupta ([11], [12], [13], [14]), Gupta and Tripathi ([15], [16])).

Gupta [17] defined the MPSD with probability function given by

$$P_1[X = x] = a(x) \frac{(g(\theta))^x}{f(\theta)}, \quad x \in T \quad (1)$$

where $a(x) > 0$ and T is a subset of the set of non-negative integers, $g(\theta)$, $f(\theta)$ are positive, finite and differentiable, θ is the parameter. In case $g(\theta)$ is invertible it reduces to Patil's [18] generalized power series distribution and if in addition T is the entire set of non-negative integers it reduces to power services distribution (PSD) given by Noack [19]. The class of distributions (1) includes among others the generalized negative binomial, generalized Poisson and Borel distributions.

Gupta [17] obtained the mean (μ_1) and variance (μ_2) of MPSD (1) given by

$$\mu_1(\theta) = \frac{g(\theta)f'(\theta)}{f(\theta)g'(\theta)} \quad (2)$$

and
$$\mu_2 = \frac{g(\theta)}{g'(\theta)} \frac{d\mu_1(\theta)}{d\theta} \quad (3)$$

A size-biased MPSD is obtained by taking the weight of MPSD (1) as X , given by

$$P_2[X = x] = \frac{xa(x)(g(\theta))^x}{\mu_1(\theta)f(\theta)} = \frac{b(x)(g(\theta))^x}{\mu_1(\theta)f(\theta)} = \frac{b(x)(g(\theta))^x}{f^*(\theta)} \quad (4)$$

where $b(x) = xa(x)$ and $f^*(\theta) = \mu_1(\theta) f(\theta)$.

As stated above we have studied the situation such that the probabilities in the distribution (4) are modified by a constant quantity α ($0 \leq \alpha \leq 1$) by increasing the probability of c value of the variable.

In this paper we are concerned with the situation where sometimes the value $c+1$ is reported erroneously as one in relation to SBMPSD. We obtain the recurrence relations between the raw moments, the central moments and the factorial moments of misclassified SBMPSD. Discussion of the effect of the misclassification on the variance is considered. To illustrate the situation under consideration some of its particular cases like the size-biased generalized negative binomial (SBGNB), the size-biased generalized Poisson (SBGP) and size-biased Borel distributions are included. Finally, an example is presented for the size-biased generalized Poisson distribution to illustrate the results.

2. MISCLASSIFIED SIZE-BIASED MODIFIED POWER SERIES DISTRIBUTION

Suppose X is a random variable having the SBMPSD (4) from which a random sample is

drawn. Assume that some of the observations corresponding to $x = c + 1$ are erroneously reported as $x = c$, and let this probability of misclassifying be α . Then the resulting distribution of X , the so called misclassified size-biased modified power series distribution (MSBMPSD), can be written in the form

$$P_3[X = x] = p(x; \alpha, \theta) = \begin{cases} g^c (b_c + \alpha b_{c+1} g) / f\mu; & x = c \\ (1 - \alpha) b_{c+1} g^{c+1} / f\mu, & x = c + 1 \\ b_{(x)} g^x / f\mu, & x \in S \end{cases} \quad (5)$$

where $S = T - (c, c+1)$ is a subset of the set I of non-negative integers not containing c and $c + 1$, $b(x)$, $f = f(\theta)$, $g = g(\theta)$, $\mu = \mu(\theta)$ are stated above and $0 \leq \alpha \leq 1$, the α being the proportion of misclassified observations. It is interesting to note that for $\alpha = 0$ the distribution (5) reduces to simple size-biased MPSD. Further if $g(\theta)$ is invertible, it reduces to the misclassified size-biased generalized power series distribution and in addition if T is regarded as an entire set I of non-negative integers, it will be called the misclassified size biased power series distribution.

In this section we obtain the mean and the variance and establish certain recurrence relations for the raw, the central and the factorial moments of the distribution (5). Here notations with * corresponds to size-biased MPSD.

2.1. Mean of the distribution. For mean, by definition, we have

$$\text{Mean} = \mu'_1 = \sum_{x \in T} x p_3(x, \alpha, \theta) = \frac{c g^c (b_c + \alpha b_{c+1} g)}{f\mu} + \frac{(c+1)(1-\alpha) b_{c+1} g^{c+1}}{f\mu} + \sum \frac{x b_x g^x}{f\mu}$$

where \sum stands for the sum over $x \in S$ here and onwards. Also, from (5) we have

$$f\mu = g^c (b_c + \alpha b_{c+1} g) + (1 - \alpha) b_{c+1} g^{c+1} + \sum b_x g^x$$

Differentiating w.r.t. θ and multiplying both sides by g/g' , we get

$$\frac{g}{g'} (f'\mu + \mu'f) = c b_c g^c + \alpha c b_{c+1} g^{c+1} + \alpha b_{c+1} g^{c+1} + (c+1)(1-\alpha) b_{c+1} g^{c+1} + \sum x b_x g^x$$

$$\frac{g}{g'} (f'\mu + \mu'f) - \alpha b_{c+1} g^{c+1} = c g^c (b_c + \alpha b_{c+1} g) + (c+1)(1-\alpha) b_{c+1} g^{c+1} + \sum x b_x g^x$$

where $\mu' = \frac{\partial}{\partial \theta} (\mu(\theta))$ and $f' = \frac{\partial}{\partial \theta} f(\theta)$ and $g' = \frac{\partial}{\partial \theta} g(\theta)$.

After simplification we get mean of (5) as

$$\begin{aligned} \text{Mean} = \mu'_1 &= \left[\frac{g}{g'} (f'\mu + \mu'f) - \alpha b_{c+1} g^{c+1} \right] / f\mu = \left(\mu + \frac{f}{f'} \frac{\partial \mu}{\partial \theta} \right) - \frac{\alpha b_{c+1} g^{c+1}}{f\mu} \\ &= \mu_1^* - (\alpha b_{c+1} g^{c+1}) / f\mu \end{aligned} \quad (6)$$

where $\mu_1^* = \mu + \frac{f}{f'} \frac{\partial \mu}{\partial \theta}$ is mean of size-biased MPSD(4).

2.2. Recurrence relation among raw moments. The r^{th} raw moment of (5) is given as

$$\begin{aligned} \mu'_r &= \sum_{x \in T} x^r p_3(x, \alpha, \theta) = c^r P[X = c] + (c+1)^r P[X = c+1] + \sum x^r P[X = x] \\ &= \frac{1}{f\mu} \left[c^r g^c (b_c + \alpha b_{c+1} g) + (c+1)(1-\alpha) b_{c+1} g^{c+1} + \sum x^r b_x g^x \right] \end{aligned} \quad (7)$$

Differentiating (7) w.r.t. θ , we get

$$\begin{aligned} \frac{\partial \mu'_r}{\partial \theta} &= \frac{1}{(f\mu)^2} \left[c^{r+1} g^{c-1} g' b_c + \alpha c^r (c+1) g^c g' b_{c+1} + (1-\alpha)(c+1)^{r+1} b_{c+1} g^c g' \right. \\ &\quad \left. + \sum x^{r+1} b_x g^{x-1} g' \right] - \left[c^r g^c (b_c + \alpha b_{c+1} g) + (1-\alpha)(c+1)^r b_{c+1} g^{c+1} \right. \\ &\quad \left. + \sum x^r b_x g^x \right] \left[\frac{f'\mu + \mu'f}{(f\mu)^2} \right] \end{aligned}$$

Multiplying both sides by g and after simplification we get

$$\begin{aligned} g \frac{\partial \mu'_r}{\partial \theta} &= g' \mu'_{r+1} + \frac{g'}{f\mu} \alpha c^r g^{c+1} b_{c+1} - \left(\frac{f'}{f} + \frac{\mu'}{\mu} \right) \mu'_r g \\ \mu'_{r+1} &= \frac{g}{g'} \frac{\partial \mu'_r}{\partial \theta} - \frac{\alpha c^r g^{c+1} b_{c+1}}{f\mu} + \left(\frac{g f'}{g' f} + \frac{g \mu'}{g' \mu} \right) \mu'_r \\ &= \frac{g}{g'} \frac{\partial \mu'_r}{\partial \theta} - \frac{\alpha c^r g^{c+1} b_{c+1}}{f\mu} + \mu_1^* \mu'_r \end{aligned}$$

This, upon using (6), reduces to

$$\mu'_{r+1} = \frac{g}{g'} \frac{\partial \mu'_r}{\partial \theta} + \alpha (\mu'_r - c^r) \frac{g^{c+1} b_{c+1}}{f\mu} + \mu_1^* \mu'_r \quad (8)$$

Higher moments can be obtained with $r = 2, 3, \dots$ etc. From (7), it is easy to establish a relation between r^{th} moment (μ'_r) of misclassified size-biased MPSD (5) and the r^{th} moment (μ^*r) of the size-biased MPSD (4) as

$$\mu'_r = \mu_r^* + \alpha (c^r - (c+1)^r) \frac{b_{c+1} g^{c+1}}{f\mu} \quad (9)$$

from which the higher order moments ($r > 1$) μ'_2, μ'_3, \dots are obtained.

2.3. Recurrence relation among central moments. The r^{th} central moment of (5) is given as

$$\begin{aligned}\mu_r &= \sum_{x \in \mathbb{T}} (x - \mu'_1)^r P_x = (c - \mu'_1)^r P_c + [(c+1) - \mu'_1]^r P_{c+1} + \sum (x - \mu'_1)^r P_x \\ &= \frac{1}{f\mu} \left[(c - \mu'_1)^r g^c (b_c + \alpha b_{c+1} g) + (1 - \alpha)(c+1 - \mu'_1)^r b_{c+1} g^{c+1} + \sum (x - \mu'_1)^r b_x g^x \right]\end{aligned}$$

Differentiating w.r.t. θ and simplifying, we get

$$\frac{\partial \mu_r}{\partial \theta} = \left(\frac{g'}{g} \right) \mu_{r+1} - r \mu_{r-1} \left(\frac{\partial \mu'_1}{\partial \theta} \right) + \alpha \left((c - \mu'_1)^r - \mu_r \right) \frac{b_{c+1} g^c g'}{f\mu}$$

This will yield

$$\mu_{r+1} = \left(\frac{g}{g'} \right) \left[\frac{\partial \mu_r}{\partial \theta} + r \mu_{r-1} \left(\frac{\partial \mu'_1}{\partial \theta} \right) \right] + \alpha b_{c+1} g^{c+1} \left[\mu_r - (c - \mu'_1)^r \right] / f\mu \quad (10)$$

Putting $r=1$ in (10) and noting $\mu_0 = 1$ and $\mu_1 = 0$, we get the variance (μ_2) of (5) as

$$\mu_2 = \frac{g}{g'} \left(\frac{\partial \mu'_1}{\partial \theta} \right) + \alpha (\mu'_1 - c) b_{c+1} \frac{g^{c+1}}{f\mu} \quad (11)$$

Using (11) in (10) we obtain a recurrence relation for central moments as

$$\begin{aligned}\mu_{r+1} &= \frac{g}{g'} \frac{\partial \mu_r}{\partial \theta} + r \mu_{r-1} \left[\mu_2 - \alpha \frac{(\mu'_1 - c) b_{c+1} g^{c+1}}{f\mu} \right] + \frac{\alpha b_{c+1} g^{c+1} \left[\mu_r - (c - \mu'_1)^r \right]}{f\mu} \\ &= \frac{g}{g'} \frac{\partial \mu_r}{\partial \theta} + r \mu_{r-1} \mu_2 + \alpha b_{c+1} g^{c+1} \frac{\left[\mu_r - (c - \mu'_1)^r - r \mu_{r-1} (\mu'_1 - c) \right]}{f\mu}\end{aligned} \quad (12)$$

where $r = 2, 3, \dots$ etc for higher order moments.

2.4. Recurrence relation among factorial moments. The r th factorial moment of (5) is given by

$$\begin{aligned}\mu'_{[r]} &= \sum_{x \in \mathbb{T}} x^{[r]} P_x = c^{[r]} P_c + (c+1)^{[r]} P_{c+1} + \sum x^{[r]} P_x \\ &= \frac{c^{[r]} g^c (b_c + \alpha b_{c+1} g)}{f\mu} + \frac{(c+1)^{[r]} (1 - \alpha) b_{c+1} g^{c+1}}{f\mu} + \frac{\sum x^{[r]} b_x g^x}{f\mu}\end{aligned} \quad (13)$$

Differentiating (13) w.r.t. θ , we get

$$\begin{aligned}\left(\frac{\partial \mu'_{[r]}}{\partial \theta} \right) &= \frac{1}{f\mu} \left[c c^{[r]} g^{c-1} g' (b_c + \alpha b_{c+1} g) + (c+1)^{[r]} (c+1) (1 - \alpha) b_{c+1} g^c g' \right. \\ &\quad \left. + \sum x x^{[r]} b_x g^{x-1} g' \right] - \mu'_{[r]} \left[\frac{f'}{f} + \frac{\mu'}{\mu} \right] + \frac{\alpha b_{c+1} c^{[r]} g^c g'}{f\mu}\end{aligned}$$

Using the identity $x \cdot x^{[r]} = x^{[r+1]} + r x^{[r]}$, we get

$$\begin{aligned}
\frac{\partial \mu'_{[r]}}{\partial \theta} &= \frac{1}{f\mu} \left[\{c^{[r+1]} + rc^{[r]}\} g^{c-1} g'(b_c + \alpha b_{c+1} g) + \{(c+1)^{[r+1]} + r(c+1)^{[r]}\} \right. \\
&\quad \left. (1-\alpha) b_{c+1} g^c g' + \sum \{x^{[r+1]} + rx^{[r]}\} b_x g^{x-1} g' \right] - \mu'_{[r]} \left[\frac{f'}{f} + \frac{\mu'}{\mu} \right] + \frac{\alpha b_{c+1} c^{[r]} g^c g'}{f\mu} \\
\mu'_{[r+1]} &= \frac{g}{g'} \frac{\partial \mu'_{[r]}}{\partial \theta} - r\mu'_{[r]} - \frac{\alpha c^{[r]} b_{c+1} g^{c+1}}{f\mu} + \mu'_{[r]} \left[\frac{f'g}{g'f} + \frac{f}{f'} \frac{\partial \mu'_{[1]}}{\partial \theta} \right] \\
&= \frac{g}{g'} \frac{\partial \mu'_{[r]}}{\partial \theta} - [\mu'_{[r]}^* - r] \mu'_{[r]} - \frac{\alpha c^{[r]} b_{c+1} g^{c+1}}{f\mu}
\end{aligned} \tag{14}$$

where $\mu'_{[1]}^*$ is first factorial moments of size biased MPSD(4).

Again in view of (6), this can be put in the form

$$\mu'_{[r+1]} = \left(\frac{g}{g'} \right) \left(\frac{\partial \mu'_{[r]}}{\partial \theta} \right) + [\mu'_{[1]} - r] \mu'_{[r]} - \frac{\alpha (\mu'_{[r]} - c^{[r]}) b_{c+1} g^{c+1}}{f\mu} \tag{15}$$

where $r = 2, 3, \dots$ and $\mu'_{[1]}, \mu'_{[2]}$ are given by

$$\mu'_{[1]} = \sum_{x \in T} x^{[1]} P_x = \sum_{x \in T} x P_x = \mu'_1$$

$$\text{and } \mu'_{[2]} = \sum_{x \in T} x^{[2]} P_x = \sum_{x \in T} x(x-1) P_x = \mu'_2 - \mu'_1$$

where μ'_1 and μ'_2 are obtained from (8) with $r = 0$ and $r = 1$

3. VARIANCE COMPARISON

In this section we discuss how the variance of (5) is effected when the reporting observations are erroneously misclassified. From (9) we have

$$\mu'_1 = \mu_1^* + \alpha(c - (c+1)) b_{c+1} g^{c+1} / f\mu \tag{16}$$

$$\text{and } \mu'_2 = \mu_2^* + \alpha(c^2 - (c+1)^2) b_{c+1} g^{c+1} / f\mu \tag{17}$$

Hence the variance of the distribution (5) is given by

$$\begin{aligned}
\mu_2 &= \mu_2^* + \alpha(c^2 - (c+1)^2) b_{c+1} g^{c+1} / f\mu - (\mu_1^* + \alpha(c - (c+1)) b_{c+1} g^{c+1} / f\mu)^2 \\
&= (\mu_2^* - \mu_1^{*2}) + \alpha(-2c-1) b_{c+1} g^{c+1} / f\mu + \alpha(2\mu_1^* - \alpha b_{c+1} g^{c+1} / f\mu) \frac{b_{c+1} g^{c+1}}{f\mu} \\
\mu_2 &= \mu_2^* + \alpha(2\mu_1^* - 2c - 1 - \alpha b_{c+1} g^{c+1} / f\mu) \frac{b_{c+1} g^{c+1}}{f\mu}
\end{aligned}$$

where $\mu_2^* = \left(\frac{g}{g'} \frac{\partial \mu_1^*}{\partial \theta} \right)$ is the variance of the size-biased MPSD (4).

This gives a relation between the variance μ_2 of (5) and the variance μ_2^* of (4) as

$$\mu_2 = \mu_2^* + \phi(\alpha, \theta) \quad (18)$$

where $\phi(\alpha, \theta)$ is a function of α and θ only given by

$$\phi(\alpha, \theta) = \alpha \left[2\mu_1^* - 2c - 1 - \alpha b_{c+1} g^{c+1} / f\mu \right] \frac{b_{c+1} g^{c+1}}{f\mu} \quad (19)$$

with

$$\mu_1^* = \left(\frac{f'g}{fg'} + \frac{f}{f'} \frac{\partial \mu(\theta)}{\partial \theta} \right) \quad \text{and} \quad \mu_2^* = \frac{g}{g'} \frac{\partial \mu_1^*}{\partial \theta}$$

The above relation between the two variances shows that the variance of the size-biased MPSD has been affected by a term $\phi(\alpha, \theta)$ and is due to reporting the observations erroneously. We note that this misclassification has reasonably moderate effect on the variance. Now if we take the ratio of both the variances we have

$$Z = \frac{\mu_2}{\mu_2^*} = 1 + \frac{\phi(\alpha, \theta)}{\mu_2^*} \quad (20)$$

This ratio Z shows that μ_2 may be equal to, greater than or less than μ_2^* depending upon the value of α and θ ; that is, on the term $\phi(\alpha, \theta)$. This term would be useful in studying the effect of misclassification on the variance. This effect can be studied in two ways

- i) When the value of θ is fixed.
- ii) When the value of α is fixed.

Here we discuss the case (i) only. Similarly case (ii) can be dealt with. Thus when the value of θ is fixed, the ratio Z will be a function of α only. That is to say $Z = \phi(\alpha)$. Hence we have

$$\frac{\partial Z}{\partial \alpha} = \frac{b_{c+1} g^{c+1} (2\mu_1^* - 2c - 1 - 2\alpha b_{c+1} g^{c+1} / f\mu)}{\mu_2^*} \quad (21)$$

$$\text{and} \quad \frac{\partial^2 Z}{\partial \alpha^2} = -2 \left(\frac{b_{c+1} g^{c+1}}{f\mu} \right)^2 / \mu_2^* < 0$$

From this, we note that the curve $Z = \phi(\alpha)$ seems to be a concave towards the origin.

Equating $\frac{\partial Z}{\partial \alpha}$ to zero gives the value of α as

$$\alpha = (2\mu_1^* - 2c - 1) / (2b_{c+1} g^{c+1} / f\mu) \quad (22)$$

and hence the maximum value of Z becomes as

$$Z_{\max} = 1 + \frac{(2\mu_1^* - 2c - 1)^2}{(4\mu_2^*)} \quad (23)$$

It is clear from (23) that the ratio Z is always greater than unity. This indicates that misclassification has reasonably a moderate effect on the variance. Further, it would be interesting to see that this ratio, when graphed on the (α, Z) axis, will provide an invertible parabola very concave to origin having its vertex at the point

$$\left[\frac{(2\mu_1^* - 2c - 1)}{(2b_{c+1}g^{c+1}/f\mu)}, 1 + \frac{(2\mu_1^* - 2c - 1)^2}{(2\sqrt{\mu_2^*})^2} \right] \quad (24)$$

when the value of θ is fixed. This parabola would be more useful in studying the behavioral effect of misclassification on the variance. Of course, one has to treat separately various possible specific cases with respect to their situations being observed in practice.

4. SOME APPLICATIONS

We illustrate here the situation under consideration defined by (5) in which some of the observations corresponding to the value $(c+1)$ are sometimes reported erroneously as c , for some of its special cases like the size-biased generalized negative binomial distribution (SBGNBD), the size-biased generalized Poisson distribution (SBGPD) and size-biased Borel distribution, and apply them to the results obtained in section 2 and 3.

4.1. Misclassified size-biased GNBD. Jain and Consul [20] defined the generalized negative binomial distribution (GNBD) as

$$P[X = x] = \frac{m\Gamma(m + \beta x)}{x!\Gamma(m + \beta x - x + 1)} \theta^x (1 - \theta)^{m + \beta x - x} ; x \in T \quad (25)$$

where $0 < \theta < 1$; $m > 0$; $|\beta\theta| < 1$

It reduces to (1) with $a(x) = \frac{m\Gamma(m + \beta x)}{x!\Gamma(m + \beta x - x + 1)}$, $g(\theta) = \theta(1 - \theta)\beta - 1$, $f(\theta) = (1 - \theta)^{-m}$

Suppose X has a size- biased GNBD given by

$$P[X = x] = \frac{\Gamma(m + \beta x)(1 - \theta\beta)\theta^{x-1}(1 - \theta)^{m + \beta x - x}}{(x - 1)!\Gamma(m + \beta x - x + 1)} ; x \in T \quad (26)$$

which is size-biased MPSD (4) with

$$b(x) = x \cdot a(x) = \frac{m\Gamma(m + \beta x)}{(x - 1)!\Gamma(m + \beta x - x + 1)} \text{ and } \mu(\theta) = \frac{m\theta}{(1 - \theta\beta)}$$

$$g(\theta) = \theta(1 - \theta)^{\beta-1}, f(\theta) = (1 - \theta)^{-m}$$

Let us assuming that the value $(c + 1)$ is sometimes reported as c in (26) and let the probability of misclassifying these observations be α . Then the resulting distribution of X , the so called a misclassified size -biased GNBD, can be defined as

$$P[x : \alpha, \theta] = \begin{cases} \frac{[\theta(1-\theta)^{\beta-1}]^c \left[\frac{m\Gamma(m+\beta c)}{(c-1)!\Gamma(m+\beta c-c+1)} + \frac{\alpha m\Gamma m + \beta(c+1)}{c!\Gamma m + \beta(c+1)-c} \theta(1-\theta)^{\beta-1} \right]}{(1-\theta)^{-m} m\theta / (1-\theta\beta)} & ; x = c \\ \frac{(1-\alpha)m\Gamma m + \beta(c+1)}{c!\Gamma m + \beta(c+1)-c} \frac{[\theta(1-\theta)^{\beta-1}]^{c+1}}{(1-\theta)^{-m} m\theta / (1-\theta\beta)} & ; x = c+1 \\ \frac{m\Gamma(m+\beta x)}{(x-1)!\Gamma(m+\beta x-x+1)} \frac{[\theta(1-\theta)^{\beta-1}]^x}{(1-\theta)^{-m} m\theta / (1-\theta\beta)} & x \in s \end{cases} \quad (27)$$

where $0 < \theta < 1$, $0 \leq \alpha \leq 1$, $m > 0$ and $|\theta\beta| < 1$.

Using the values of $b(x)$, $\mu(\theta)$, $g(\theta)$ and $f(\theta)$ in the foregoing results we obtain results for (27) as under

$$4.1.1. \text{ Mean} = \mu'_1 = \frac{m\theta}{(1-\theta\beta)} + \frac{(1-\theta)}{(1-\theta\beta)^2} - \frac{\alpha}{m} b_{c+1} g^c (1-\theta)^{\beta+m-1} (1-\theta\beta) \quad (28)$$

$$4.1.2. \text{ Variance} = \mu_2 = \frac{\theta(1-\theta)}{(1-\theta\beta)^4} [m - m\theta\beta - \theta\beta + 2\beta - 1] + \frac{\alpha}{m} b_{c+1} g^c \\ \left[\frac{2m\theta(1-\theta\beta) + 2(1-\theta) - (2c+1)(1-\theta\beta)^2 - \frac{\alpha}{m} b_{c+1} g^c (1-\theta)^{\beta+m-1} (1-\theta\beta)^3}{(1-\theta)^{\beta+m-1} (1-\theta\beta)^{-1}} \right] \quad (29)$$

4.1.3. Recurrence relation among raw moments

$$\mu'_{r+1} = \theta(1-\theta)(1-\theta\beta)^{-1} \frac{\partial \mu'_r}{\partial \theta} + \alpha(\mu'_r - c^r) m^{-1} b_{c+1} g^c (1-\theta\beta)(1-\theta)^{\beta+m-1} + \mu'_1 \mu'_r \quad (30)$$

4.1.4. Recurrence relation among central moments

$$\mu_{r+1} = \theta(1-\theta)(1-\theta\beta)^{-1} \frac{\partial \mu_r}{\partial \theta} + r\mu_{r-1} \mu_2 + \alpha b_{c+1} g^c m^{-1} (1-\theta\beta)(1-\theta)^{\beta+m-1} \\ \left[\mu_r - (c - \mu'_1)^r - r\mu_{r-1} (\mu'_1 - c) \right] \quad (31)$$

4.1.5. Recurrence relation among factorial moments.

$$\mu'_{[r+1]} = \theta(1-\theta)(1-\theta\beta)^{-1} \frac{\partial \mu'_{[r]}}{\partial \theta} + [\mu'_{[1]} - r] \mu'_{[r]} - \alpha b_{c+1} g^c m^{-1} \\ (1-\theta\beta)(1-\theta)^{\beta+m-1} [\mu'_{[r]} - c^{[r]}] \quad (32)$$

4.1.6. **The variance ratio** $Z = \frac{\mu_2}{\mu_2^*}$ **becomes**

$$Z = 1 + \frac{\frac{\alpha}{m} b_{c+1} g^c (1-\theta)^{\beta+m-2} (1-\theta\beta)^3 \left[\begin{array}{l} 2m\theta(1-\theta\beta) + 2(1-\theta) - (2c+1)(1-\theta\beta)^2 \\ -\frac{\alpha}{m} b_{c+1} g^c (1-\theta)^{\beta+m-1} (1-\theta\beta)^3 \end{array} \right]}{\theta[m - m\theta\beta - \theta\beta + 2\beta - 1]} \quad (33)$$

For a fixed value of θ , the value α will be

$$\alpha = \frac{m[2m\theta(1-\theta\beta) + 2(1-\theta) - (2c+1)(1-\theta\beta)^2]}{2(1-\theta\beta)^3 b_{c+1} g^c (1-\theta)^{\beta+m-1}} \quad (34)$$

for which the maximum value Z comes out as

$$Z_{\max} = 1 + \frac{[2m\theta(1-\theta\beta) + 2(1-\theta) - (2c+1)(1-\theta\beta)^2]^2}{4\theta(1-\theta)(m + 2\beta - 1 - m\theta\beta - \theta\beta)} \quad (35)$$

The invertible parabola when graphed on (α, Z) axis will have the vertex at the point

$$\frac{m[2m\theta(1-\theta\beta) + 2(1-\theta) - (2c+1)(1-\theta\beta)^2]}{2(1-\theta\beta)^3 b_{c+1} g^c (1-\theta)^{\beta+m-1}}; \\ 1 + \frac{[2m\theta(1-\theta\beta) + 2(1-\theta) - (2c+1)(1-\theta\beta)^2]^2}{4\theta(1-\theta)(m + 2\beta + m\theta\beta - \theta\beta)} \quad (36)$$

4.2. **Misclassified size-biased Generalized Poisson distribution.** Consul and Jain [21] defined the generalized Poisson distribution (GPD) as

$$P[X = x] = \frac{\lambda_1 (\lambda_1 + \lambda_2 x)^{x-1} e^{-(\lambda_1 + \lambda_2 x)}}{x!} \quad x = 0, 1, 2, \dots \quad (37)$$

Shoukri and Consul [22] modified the form of (37) to

$$P[X = x] = \frac{(1 + \beta x)^{x-1} \theta^x e^{-\theta(1+\beta x)}}{x!}; \quad \theta > 0, |\theta\beta| < 1, x = 0, 1, 2, \dots \quad (38)$$

where $\theta = \lambda_1$ and $\beta = \frac{\lambda_2}{\lambda_1}$

It is a particular case of Gupta [17] MPSD with

$$a(x) = \frac{(1 + \beta x)^{x-1}}{x!}, \quad g(\theta) = \theta e^{-\theta\beta}, \quad f(\theta) = e^\theta$$

Now suppose X has a size-biased GPD given by

$$P[X = x] = \frac{(1 + \beta x)^{x-1} (1 - \theta\beta)\theta^{x-1} e^{-\theta(\beta x+1)}}{(x-1)!}; \quad x = 1, 2, \dots \quad (39)$$

which is size-biased MPSD (4) with

$$b(x) = xa(x) = \frac{(1 + \beta x)^{x-1}}{(x-1)!}, \quad \mu(\theta) = \frac{\theta}{(1 - \theta\beta)}, \quad f(\theta) = e^\theta, \quad g(\theta) = \theta e^{-\theta\beta}$$

Assume that some of the values $c + 1$ are erroneously reported as c and let α be the probability of misclassifying them. Then the resulting distribution of X , the so called misclassified size-biased GPD, can be defined as

$$P[X = x] = \begin{cases} \frac{(\theta e^{-\theta\beta})^c \left[\frac{(1 + \beta c)^{c-1}}{(c-1)!} + \frac{\alpha [1 + \beta(c+1)]^c (\theta e^{-\theta\beta})}{c!} \right]}{\frac{e^\theta \theta}{(1 - \theta\beta)}}; & x = c \\ \frac{(1 - \alpha) [1 + \beta(c+1)]^c (\theta e^{-\theta\beta})^{c+1}}{c! \theta e^\theta / (1 - \theta\beta)} & x = c + 1 \\ \frac{(1 + \beta x)^{x-1} (\theta e^{-\theta\beta})^x}{(x-1)! \theta e^\theta / (1 - \theta\beta)} & x \in S \end{cases} \quad (40)$$

where S is defined earlier; $0 \leq \alpha \leq 1$, $\theta > -1$ and $|\beta\theta| < 1$.

Using the values of $b(x)$, $\mu(\theta)$, $g(\theta)$ and $f(\theta)$ in the forgoing results we obtain the results for (40):

$$4.2.1. \text{ Mean } = \mu'_1 = \frac{1}{(1 - \theta\beta)^2} + \frac{\theta}{(1 - \theta\beta)} - \alpha(1 - \theta\beta)b_{c+1}g^c e^{-\theta(\beta+1)} \quad (41)$$

$$4.2.2. \mu_2 = \frac{(2\beta\theta - \theta^2\beta + \theta)}{(1 - \theta\beta)^4} + \alpha \left[2 + 2\theta(1 - \theta\beta) - (2c + 1)(1 - \theta\beta)^2 - \alpha b_{c+1}g^c \right. \\ \left. (1 - \theta\beta)^3 e^{-\theta(\beta+1)} \right] [b_{c+1}g^c (1 - \theta\beta)^{-1} e^{-\theta(\beta+1)}] \quad (42)$$

4.2.3. **Recurrence relation among raw moments**

$$\mu'_{r+1} = \frac{\theta}{(1 - \theta\beta)} \frac{\partial \mu'_r}{\partial \theta} + \alpha b_{c+1}g^c (1 - \theta\beta) e^{-\theta(\beta+1)} [\mu'_r - c^r] + \mu'_1 \mu'_r \quad (43)$$

4.2.4. **Recurrence relation among central moments**

$$\mu_{r+1} = \frac{\theta}{(1 - \theta\beta)} \left(\frac{\partial \mu_r}{\partial \theta} \right) + r\mu_2 \mu_{r-1} + \alpha b_{c+1}g^c (1 - \theta\beta) \\ \left[\mu_r - (c - \mu'_1)^r - r\mu_{r-1} (\mu'_1 - c) e^{-\theta(\beta+1)} \right] \quad (44)$$

4.2.5. **Recurrence relation among factorial moments**

$$\mu'_{[r+1]} = \frac{\theta}{(1 - \theta\beta)} \frac{\partial \mu'_{[r]}}{\partial \theta} + [\mu'_{[1]} - r] \mu'_{[r]} - \alpha b_{c+1}g^c (1 - \theta\beta) [\mu'_{[r]} - c^{[r]}] e^{-\theta(\beta+1)} \quad (45)$$

4.2.6. **The variance ratio** $Z = \frac{\mu_2}{\mu_2^*}$ **becomes**

$$Z = 1 + \alpha \left[2 + 2\theta(1 - \theta\beta) - (2c + 1)(1 - \theta\beta)^2 - \alpha b_{c+1} g^c (1 - \theta\beta)^3 e^{-\theta(\beta+1)} \right] \\ \times b_{c+1} g^c e^{-\theta(1+\beta)} (1 - \theta\beta)(2\beta\theta - \theta^2\beta + \theta)^{-1} \quad (46)$$

For the fixed value of θ , the value of α is

$$\alpha = \left[2 + 2\theta(1 - \theta\beta) - (2c + 1)(1 - \theta\beta)^2 \right] / 2b_{c+1} g^c e^{-\theta(1+\beta)} (1 - \theta\beta)^3 \quad (47)$$

for which the maximum value Z will be

$$Z_{\max} = 1 + \frac{\left[2 + 2\theta(1 - \theta\beta) - (2c + 1)(1 - \theta\beta)^2 \right]^2}{4(2\beta\theta + \theta - \theta^2\beta)} \quad (48)$$

The invertible parabola when graphed on (α, Z) axis will have the vertex at the point

$$\left[\frac{2 + 2\theta(1 - \theta\beta) - (2c + 1)(1 - \theta\beta)^2}{2b_{c+1} g^c e^{-\theta(1+\beta)} (1 - \theta\beta)^3}, 1 + \frac{\left[2 + 2\theta(1 - \theta\beta) - (2c + 1)(1 - \theta\beta)^2 \right]^2}{4[2\beta\theta + \theta - \theta^2\beta]} \right] \quad (49)$$

4.3. **Misclassified Size-biased Borel distribution.** Borel [23] has defined a Borel distribution with one parameter for the random variable X as

$$P[X = x] = p(x, \theta) = \frac{(1 + x)^{x-1}}{(x)!} \theta^x e^{-(1+x)\theta}; \quad 0 < \theta < 1, \quad x = 1, 2, \dots \quad (50)$$

(This distribution describes a distribution of the number of customers served before a queue vanishes under condition of a single queue with random arrival times (at constant rate) of customers and a constant time occupied in serving each customer).

It is a particular case of modified power series distribution (1) with

$$a(x) = \frac{(1 + x)^{x-1}}{x!}, \quad g(\theta) = (\theta e^{-\theta}), \quad f(\theta) = e^{\theta}$$

Now suppose X has a size-biased Borel distribution with p.d.f given by

$$P[X = x] = \frac{(1 + x)^{x-1}}{(x-1)!} \theta^{x-1} (1 - \theta) e^{-\theta(1+x)}; \quad x = 1, 2, \dots \quad (51)$$

which is size-biased MPSD (4) with

$$b(x) = x \cdot a(x) = \frac{(1 + x)^{x-1}}{(x-1)!}, \quad \mu(\theta) = \frac{\theta}{1 - \theta}, \quad g(\theta) = \theta e^{-\theta}, \quad f(\theta) = e^{\theta}$$

Assume that some of the values $(c + 1)$ are erroneously reported as c , and let the proportions of these observations be α . Then the resulting distribution of X , the so called misclassified size-biased Borel distribution can be defined as

$$P[X = x] = \begin{cases} (\theta e^{-\theta})^c \left[\frac{(1+c)^{c-1}}{(c-1)!} + \frac{\alpha(2+c)^c (\theta e^{-\theta})}{c!} \right] / \frac{\theta e^{-\theta}}{(1-\theta)} & x = c \\ \frac{(1-\alpha)[c+2]^c (\theta e^{-\theta})^{c+1}}{c! \theta e^{-\theta} / (1-\theta)} & x = c+1 \\ \frac{(1+x)^{x-1} (\theta e^{-\theta})^x}{(x-1)! \theta e^{-\theta} / (1-\theta)} & x \in S \end{cases} \quad (52)$$

where S is defined earlier, $0 \leq \alpha \leq 1$, $0 < \theta < 1$ and $x = 1, 2, 3, \dots$

Using values of $b(x)$, $\mu(\theta)$, $g(\theta)$ and $f(\theta)$ in the forgoing results we obtain the results for (52) as

$$4.3.1. \text{ Mean} = \mu'_1 = \frac{1}{(1-\theta)^2} + \frac{\theta}{(1-\theta)} - \alpha(1-\theta)b_{c+1}g^c e^{-2\theta} \quad (53)$$

$$4.3.2. \text{ Variance} = \mu_2 = \frac{(3\theta - \theta^2)}{(1-\theta)^4} + \alpha \left[2 + 2\theta(1-\theta) - (2c+1)(1-\theta)^2 - \alpha b_{c+1}g^c \right. \\ \left. (1-\theta)^3 e^{-2\theta} \right] [b_{c+1}g^c (1-\theta)^{-1} e^{-2\theta}] \quad (54)$$

4.3.3. **Recurrence relation among raw moments**

$$\mu'_{r+1} = \frac{\theta}{(1-\theta)} \frac{\partial \mu'_r}{\partial \theta} + \alpha b_{c+1}g^c (1-\theta)e^{-2\theta} [\mu'_r - c^r] + \mu'_1 \mu'_r \quad (55)$$

4.3.4. **Recurrence relation among central moments**

$$\mu_{r+1} = \frac{\theta}{(1-\theta)} \frac{\partial \mu'_r}{\partial \theta} + r\mu_2 \mu_{r-1} + \alpha b_{c+1}g^c (1-\theta) [\mu_r - (c - \mu'_1)^r - r\mu_{r-1}(\mu'_1 - c)e^{-2\theta}] \quad (56)$$

4.3.5. **Recurrence relation among factorial moments**

$$\mu'_{[r+1]} = \frac{\theta}{(1-\theta)} \frac{\partial \mu'_{[r]}}{\partial \theta} + [\mu'_{[1]} - r] \mu'_{[r]} - \alpha b_{c+1}g^c (1-\theta) [\mu'_{[r]} - c^{[r]}] e^{-2\theta} \quad (57)$$

4.3.6 **The variance ratio** $Z = \frac{\mu_2}{\mu_2^*}$ **will be**

$$Z = 1 + \alpha \left[2 + 2\theta(1-\theta) - (2c+1)(1-\theta)^2 - \alpha b_{c+1}g^c (1-\theta)^3 e^{-2\theta} \right] b_{c+1}g^c e^{-2\theta} \\ (1-\theta)^3 (3\theta - \theta^2 \beta)^{-1} \quad (58)$$

For fixed value of θ , the value of α is

$$\alpha = \left[2 + 2\theta(1-\theta) - (2c+1)(1-\theta)^2 \right] / 2b_{c+1}g^c e^{-2\theta} (1-\theta)^3 \quad (59)$$

for which the maximum value Z will be

$$Z_{\max} = 1 + \frac{[2 + 2\theta(1 - \theta) - (2c + 1)(1 - \theta)^2]^2}{4(3\theta - \theta^2)} \quad (60)$$

The invertible parabola when graphed on (α, Z) axis will have the vertex at the point

$$\left[\frac{2 + 2\theta(1 - \theta) - (2c + 1)(1 - \theta)^2}{2b_{c+1}g^c e^{-2\theta}(1 - \theta)^3} \right]; 1 + \frac{[2 + 2\theta(1 - \theta) - (2c + 1)(1 - \theta)^2]^2}{4[3\theta - \theta^2]} \quad (61)$$

5. SOME NUMERICAL RESULTS

To see the effect of misclassification on variance a case of misclassified size-biased Borel distribution and misclassified size-biased GNB distribution with $m=4$ and $\beta=1$ is considered. It is assumed that some of the observations, which correspond to the value two, are reported erroneously as value one. The mean and the variances of the misclassified size-biased Borel distribution and misclassified size-biased GNBD have been calculated for different values of α and θ and tabulated in the Tables 1 and 2 respectively, which shows the comparative study of the change in the mean and the variance of the distribution due to a change in α , i.e. the proportion of misclassifying the observation corresponding to $x = 2$ by reporting it as one corresponding to $x = 1$. The values in the first row of these two tables corresponding to $\alpha = 0$ are the values of the mean and the variance of the non-misclassified or usual distributions i.e. size-biased Borel distribution and Size-biased GNBD respectively. The tables shows that as α increases, the mean of the distribution decreases while the variance decreases for small values of θ , while it increases for large values of θ . This can also be seen from the Tables 3 and 4, which gives the values of the ratio $(Z = \frac{\mu_2}{\mu_2^*})$ of the variance of the misclassified distribution to that of the usual distribution.

TABLE 1. The Values of Mean and Variance for Misclassified Size-Biased Borel Distribution

	$\theta = 0.1$		$\theta = 0.5$		$\theta = 0.9$	
	μ'_1	μ_2	μ'_1	μ_2	μ'_1	μ_2
$\alpha = 0.0$	1.3457	0.4420	5.0000	20.0000	109.0000	18900.0000
$\alpha = 0.1$	1.3257	0.4354	4.9832	20.1170	108.9982	18900.3901
$\alpha = 0.5$	1.2457	0.4011	4.9163	10.5787	108.9909	18901.9505
$\alpha = 0.9$	1.1657	0.3540	4.8494	21.0316	108.9837	18903.5108
$\alpha = 1.0$	1.1457	0.3403	4.8326	21.1434	108.9818	18903.9001

TABLE 2. The Values of Mean and Variance for Misclassified Size-Biased Generalized Negative Binomial Distribution with $m=4, \beta=1$

	$\theta = 0.1$		$\theta = 0.5$		$\theta = 0.9$	
	μ'_1	μ_2	μ'_1	μ_2	μ'_1	μ_2
$\alpha = 0.0$	0.4444	0.6173	4.0000	10.0000	36.0000	450.0000
$\alpha = 0.1$	0.4149	0.6152	3.9922	10.0702	35.9999	450.0004
$\alpha = 0.5$	0.2968	0.6119	3.9609	10.3500	35.9999	450.0020
$\alpha = 0.9$	0.1787	0.5762	3.9297	10.6279	35.9999	450.0036
$\alpha = 1.0$	0.1492	0.5629	3.9219	10.6970	35.9999	450.0040

TABLE 3. Variance Ratio ($Z = \frac{\mu_2}{\mu_2^*}$) Values for Size-Biased Borel Distribution

	$\theta=0.05$	$\theta=0.1$	$\theta=0.3$	$\theta=0.5$	$\theta=0.7$	$\theta=0.9$
$\alpha=0.1$	0.9532	0.9851	1.0145	1.0058	1.0009	1.00001
$\alpha=0.3$	0.8546	0.9499	1.0424	1.0174	1.0028	1.00004
$\alpha=0.5$	0.7494	0.9075	1.0687	1.0289	1.0046	1.00007
$\alpha=0.7$	0.6375	0.8578	1.0935	1.0403	1.0065	1.00010
$\alpha=0.9$	0.5190	0.8009	1.1167	1.0516	1.0083	1.00013
$\alpha=1.0$	0.4573	0.7700	1.1277	1.0572	1.0092	1.00015

TABLE 4. Variance Ratio ($Z = \frac{\mu_2}{\mu_2^*}$) Values for GNBD with $m=4$ and $\beta = 1$

	$\theta=0.05$	$\theta=0.1$	$\theta=0.3$	$\theta=0.5$	$\theta=0.7$	$\theta=0.9$
$\alpha=0.1$	0.9656	1.0039	1.0269	1.0070	1.0005	1.0000
$\alpha=0.3$	0.8886	1.0032	1.0793	1.0210	1.0015	1.0000
$\alpha=0.5$	0.8008	0.9913	1.1301	1.0350	1.0024	1.0000
$\alpha=0.7$	0.7023	0.9680	1.1792	1.0489	1.0034	1.0000
$\alpha=0.9$	0.5929	0.9334	1.2267	1.0638	1.0044	1.0000
$\alpha=1.0$	0.5341	0.9119	1.2498	1.0697	1.0049	1.0000

6. AN ILLUSTRATIVE EXAMPLE (GOODNESS OF FIT)

To illustrate the practical application of results obtained in this paper, an example considered by Mishra and Singh [24] has been suitably altered. The example chosen is a

distribution of number of papers in Review of Applied Entomology, Vol.I (1913) observed by Williams [25]. Mishra and Singh [24] has shown that size-biased GPD gives a satisfactory good fit to the unaltered data. For our purpose of illustration it has been assumed that ten of the observations which correspond to the value two are reported erroneously as value one. We fitted the misclassified size-biased GPD (40) to the altered data. We considered θ known and obtained moment estimators of α (probability of misclassification) and β . Both the original and the altered data together with expected values of the altered data are given in Table 5.

TABLE 5. Number of authors (f_x) according to number of papers (X) in the Review of Applied Entomology, Vol.I (1913) (Williams [25])

X	Original sample frequencies	Altered data Observed frequencies	Expected frequencies
1	285	295	298.56
2	70	60	55.20
3	32	32	27.97
4	10	10	13.72
5	4	4	7.15
6	3	3	3.84
7	3	3	2.10
8	1	1	1.17
9	2	2	0.70
10	1	1	0.59
Total	411.0	411.0	411.0
d.f			3
χ^2			3.741
$P(\chi^2)$			0.30

For the altered (misclassified data), we have $\bar{X} = 1.572$ and $S^2 = 1.581$, $\hat{\alpha} = 0.124$ and $\hat{\beta} = -0.9855$, $\theta = -0.3$ (known)

The estimate $\hat{\beta} = -0.9855$, obtained by using the altered data is to be compared with $\hat{\beta} = -0.981407$ obtained by Mishra and Singh [24], when calculations are based on the original unaltered sample. The estimate $\hat{\alpha} = 0.124$ is to be compared with $10/70 = 0.142$, which is the proportion of twos that were misclassified in the process of altering the original data for this illustration.

As indicated by the low value obtained by χ^2 , the agreement between observed and expected frequencies in the altered sample is satisfactory. With 3 degrees of freedom, $P[\chi^2 > 3.741] = 0.30$. The value $\chi^2_{(3)} = 3.741$ for the altered sample is to be

compared with $\chi^2_{(3)}=3.888$ obtained by Mishra and Singh[24], in comparing frequencies of the original sample with expected frequencies based on the unaltered sample estimate of θ and β .

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