

EXPLICIT ERROR BOUND FOR QUADRATIC SPLINE APPROXIMATION OF CUBIC SPLINE

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ABSTRACT. In this paper we find an explicit form of upper bound of Hausdorff distance between given cubic spline curve and its quadratic spline approximation. As an application the approximation of offset curve of cubic spline curve is presented using our explicit error analysis. The offset curve of quadratic spline curve is exact rational spline curve of degree six, which is also an approximation of the offset curve of cubic spline curve.

1. INTRODUCTION

Cubic and quadratic spline curves and their offset curves are most widely used in CAD/CAM or CAGD [6, 10, 13]. While offset curve of cubic spline cannot be expressed by polynomial or rational spline curve amenable to CAD/CAM system [12, 18, 24, 25], offset curve of quadratic spline can be exactly expressed by rational spline curve of degree six [9, 14, 21, 23]. This is one of the important reason why the quadratic approximation of cubic spline curves is needed [15, 20, 21].

The quadratic approximation of cubic spline is easy, but the calculation of the distance between cubic curve and its quadratic approximation curve is not [1, 4]. As the error measurement method, the Hausdorff distance between two curves is generally used in CAD/CAM or CAGD. The definition of Hausdorff distance between two curves $\mathbf{p}(s)$, $s \in [a, b]$ and $\mathbf{q}(t)$, $t \in [c, d]$, is

$$d_H(\mathbf{p}, \mathbf{q}) = \max\left\{ \max_{s \in [a, b]} \min_{t \in [c, d]} |\mathbf{p}(s) - \mathbf{q}(t)|, \max_{t \in [c, d]} \min_{s \in [a, b]} |\mathbf{p}(s) - \mathbf{q}(t)| \right\}.$$

(For more knowledge about the Hausdorff distance, refer to [5, 7, 8, 16, 17, 19].) But, it is not easy to find the Hausdorff distance between cubic Bézier curve and its quadratic Bézier

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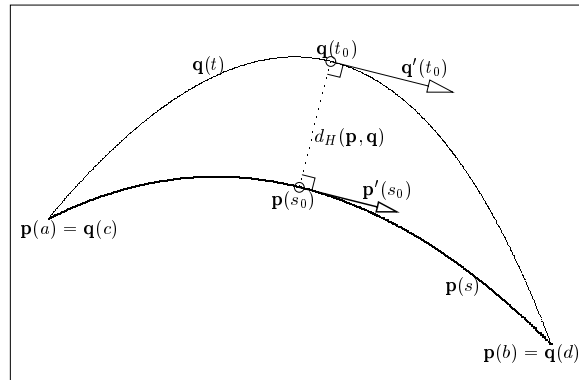


FIGURE 1. Hausdorff distance between two curves $\mathbf{p}(s)$, $s \in [a, b]$, (thick lines) and $\mathbf{q}(t)$, $t \in [c, d]$ (thin lines).

approximation. The Hausdorff distance between two differentiable curves $\mathbf{p}(s)$ and $\mathbf{q}(t)$ can be obtained by searching the points $\mathbf{p}(s_0)$ and $\mathbf{q}(t_0)$ satisfying

$$\mathbf{p}'(s_0) \circ (\mathbf{p}(s_0) - \mathbf{q}(t_0)) = 0 \quad \text{and} \quad \mathbf{q}'(t_0) \circ (\mathbf{p}(s_0) - \mathbf{q}(t_0)) = 0 \quad (1.1)$$

when one of them is admissible to the other[11], as shown in Figure 1. Thus to find the Hausdorff distance requires solving the nonlinear system of two variables such as in Equation (1.1). Although \mathbf{p} is cubic \mathbf{q} is quadratic, Equation (1.1) cannot be solved symbolically.

In this paper we present an explicit upper bound of Hausdorff distance between cubic Bézier curve and its quadratic approximation curve. As an application, we give an approximation of offset curve of cubic Bézier curves. We approximate the outline of the font 'S' consisting of cubic Bézier curves by quadratic spline curve, and we find the offset curve of the quadratic spline. The offset curve is a rational spline of degree six and also an approximation of offset curve of the cubic spline.

In §2, we present the upper bound of Hausdorff distance of the between cubic Bézier curve and the quadratic approximation. In §3, we applied our analysis to an numerical example, the quadratic approximation of cubic spline and calculation of the exact offset curve of quadratic spline in rational spline of degree six.

2. ERROR BOUND ANALYSIS FOR QUADRATIC APPROXIMATION OF CUBIC CURVE

In this section we present an error bound analysis for quadratic G^1 end-points interpolation of cubic Bézier curve. Let $\mathbf{q}(t)$ be the quadratic Bézier curve with the control points \mathbf{q}_0 , \mathbf{q}_1 and \mathbf{q}_2 . By definition of Bézier curve[13],

$$\mathbf{q}(t) = \sum_{i=0}^2 \mathbf{q}_i B_i^2(t) \quad t \in [0, 1]$$

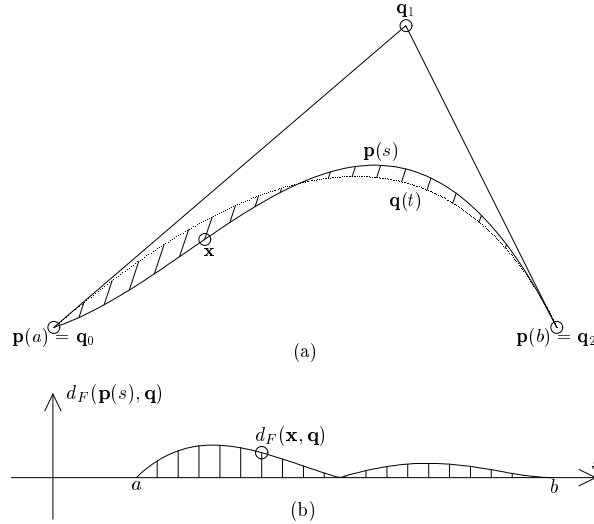


FIGURE 2. (a) Quadratic Bézier curve $\mathbf{q}(t)$ and planar curve $\mathbf{p}(s)$, $s \in [a, b]$.
 (b) $d_F(\mathbf{p}(s), \mathbf{q})$ and $d_F(\mathbf{x}, \mathbf{q})$.

where $B_i^n(t)$ is the Bernstein polynomial of degree n ,

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}.$$

Any point \mathbf{x} in the (closed) triangle $\mathbf{q}_0\mathbf{q}_1\mathbf{q}_2$ can be written uniquely in terms of barycentric coordinates τ_0, τ_1, τ_2 with respect to $\triangle\mathbf{q}_0\mathbf{q}_1\mathbf{q}_2$, where $\tau_0 + \tau_1 + \tau_2 = 1$ and $0 \leq \tau_0, \tau_1, \tau_2 \leq 1$,

$$\mathbf{x} = \tau_0\mathbf{q}_0 + \tau_1\mathbf{q}_1 + \tau_2\mathbf{q}_2.$$

Thus any function defined on $\triangle\mathbf{q}_0\mathbf{q}_1\mathbf{q}_2$ can be expressed as a function of τ_0, τ_1, τ_2 . Using the function $f : \triangle\mathbf{q}_0\mathbf{q}_1\mathbf{q}_2 \rightarrow \mathbb{R}$ defined[13] by

$$f(\mathbf{x}) = 4\tau_0\tau_2 - \tau_1^2. \tag{2.1}$$

Floater[17] presented a formula for an upper bound of the Hausdorff distance between the planar curve contained in $\triangle\mathbf{q}_0\mathbf{q}_1\mathbf{q}_2$ and the conic approximation having control points \mathbf{q}_i , $i = 0, 1, 2$, using Equation (2.1). By the restriction $w = 1$ on the conic, we have the following inequality.

Lemma 2.1. *For any continuous curve $\mathbf{p}(s)$, $s \in [a, b]$, contained in $\triangle\mathbf{q}_0\mathbf{q}_1\mathbf{q}_2$, the Hausdorff distance between $\mathbf{p}(s)$ and the quadratic Bézier curve $\mathbf{q}(t)$ is*

$$d_H(\mathbf{p}, \mathbf{q}) \leq \frac{1}{4} \max_{s \in [a, b]} |f(\mathbf{p}(s))| |\mathbf{q}_0 - 2\mathbf{q}_1 + \mathbf{q}_2|. \tag{2.2}$$

Proof. See Lemma 3.2 in Floater [17]. □

In this paper we denote the upper bound in equation above by $d_F(\mathbf{p}, \mathbf{q})$.

Remark 2.2. For any point \mathbf{x} in the triangle $\triangle \mathbf{q}_0 \mathbf{q}_1 \mathbf{q}_2$, $d_F(\mathbf{x}, \mathbf{q}) = \frac{1}{4} |f(\mathbf{x})| |\mathbf{q}_0 + \mathbf{q}_2 - 2\mathbf{q}_1|$ is the distance from the point \mathbf{x} to the curve $\mathbf{q}(t)$ in direction of $\mathbf{q}_0 + \mathbf{q}_2 - 2\mathbf{q}_1$, as shown in Figure 2, and

$$d_F(\mathbf{p}, \mathbf{q}) = \max_{s \in [a, b]} d_F(\mathbf{p}(s), \mathbf{q}).$$

We find the exact form of the distance $d_F(\mathbf{c}, \mathbf{q})$ between planar cubic Bézier curve \mathbf{c} and its quadratic G^1 end-points interpolation \mathbf{q} . Let $\mathbf{c}(s)$ be the planar cubic Bézier curve with the control points $\mathbf{c}_i, i = 0, \dots, 3$,

$$\mathbf{c}(s) := \sum_{i=0}^3 \mathbf{c}_i B_i^3(s)$$

Let $\mathbf{q}(t)$ be the G^1 end points interpolation of $\mathbf{c}(s)$, and $\mathbf{c}(s)$ be contained in the triangle $\triangle \mathbf{q}_0 \mathbf{q}_1 \mathbf{q}_2$, as shown in Figure 3. If such a quadratic Bézier curve $\mathbf{q}(t)$ does not exist or the cubic Bézier curve $\mathbf{c}(s)$ cannot be contained in $\triangle \mathbf{q}_0 \mathbf{q}_1 \mathbf{q}_2$, then $\mathbf{c}(s)$ may be subdivided at inflection points or farthest points from the line $\overline{\mathbf{q}_0 \mathbf{q}_2}$. (Refer to [1]) Thus the cubic Bézier curve $\mathbf{c}(s)$ may be expressed as

$$\begin{aligned} \mathbf{c}(s) = & \mathbf{q}_0 B_0^3(s) + ((1 - \delta_0)\mathbf{q}_0 + \delta_0 \mathbf{q}_1) B_1^3(s) + ((1 - \delta_1)\mathbf{q}_2 + \delta_1 \mathbf{q}_1) B_2^3(s) \\ & + \mathbf{q}_2 B_3^3(s) \end{aligned} \tag{2.3}$$

for some $0 \leq \delta_0, \delta_1 \leq 1$, as shown in Figure 3. Also we can represent the cubic Bézier curve $\mathbf{c}(s)$ as follows

$$\begin{aligned} \mathbf{c}(s) = & (B_0^3(s) + (1 - \delta_0)B_1^3(s))\mathbf{q}_0 + (\delta_0 B_1^3(s) + \delta_1 B_2^3(s))\mathbf{q}_1 \\ & + (B_3^3(s) + (1 - \delta_1)B_2^3(s))\mathbf{q}_2. \end{aligned}$$

From Equation (2.1), we have the equation of $f(\mathbf{c}(s))$ as

$$f(\mathbf{c}(s)) = 4(B_0^3(s) + (1 - \delta_0)B_1^3(s))(B_3^3(s) + (1 - \delta_1)B_2^3(s)) - (\delta_0 B_1^3(s) + \delta_1 B_2^3(s))^2 \tag{2.4}$$

which is a polynomial of degree six. Fortunately, we have the explicit error bound of the Hausdorff distance between cubic Bézier curve and its quadratic approximation curve as follows.

Proposition 2.3. Let $\mathbf{q}(t)$ be the quadratic G^1 end-points interpolation of cubic Bézier curve $\mathbf{c}(s)$, and $\mathbf{c}(s)$ be contained in the triangle $\triangle \mathbf{q}_0 \mathbf{q}_1 \mathbf{q}_2$. Then the upper bound of the Hausdorff distance between two curves \mathbf{c} and \mathbf{q} is given by

$$d_F(\mathbf{c}, \mathbf{q}) = \frac{1}{4} \max_{0 < \sigma_i < 1} |f(\mathbf{c}(\sigma_i))| |\mathbf{q}_0 + \mathbf{q}_2 - 2\mathbf{q}_1|$$

where $\sigma_i, i = 1, 2, 3$, are roots in $(0, 1)$ of cubic equation

$$\begin{aligned} [3(\delta_0 + \delta_1) - 4]^2 s^3 - [3(\delta_0 + \delta_1) - 4][7\delta_0 + 2\delta_1 - 6]s^2 \\ + [15\delta_0^2 + 9\delta_0\delta_1 - 18\delta_0 + 2\delta_1]s + [4(1 - \delta_1) - 3\delta_0^2] = 0. \end{aligned} \tag{2.5}$$

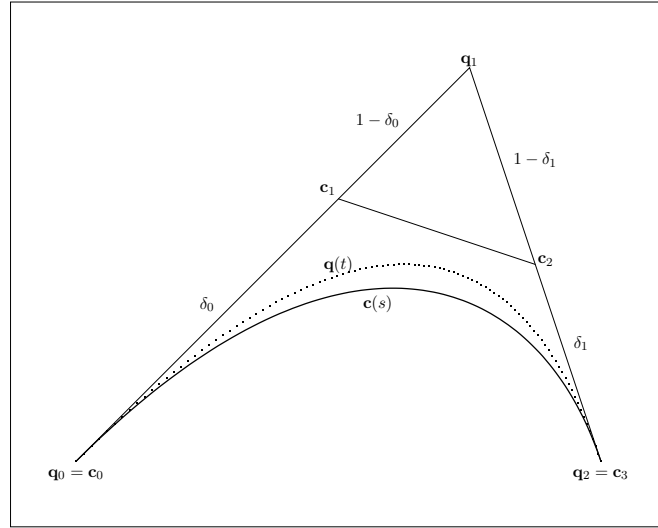


FIGURE 3. Quadratic Bézier curve $\mathbf{q}(t)$ (dotted line) with control points \mathbf{q}_0 , \mathbf{q}_1 and \mathbf{q}_2 , and cubic Bézier curve $\mathbf{c}(s)$ (solid line) with control points \mathbf{c}_i , $i = 0, \dots, 3$.

Proof. Since $f(\mathbf{c}(0)) = 0$ and $f(\mathbf{c}(1)) = 0$, $|f(\mathbf{c}(s))|$ has the maximum value at stationary points for some $s \in (0, 1)$. Therefore it is sufficient to find the roots of

$$\frac{d(f(\mathbf{c}(s)))}{ds} = 0.$$

By Equation (2.4) and the equation $B_i^n(t)B_j^m(t) = \frac{\binom{n}{i}\binom{m}{j}}{\binom{n+m}{i+j}}B_{i+j}^{n+m}(t)$ (refer to [2, 3]), we have

$$\begin{aligned} f(\mathbf{c}(s)) &= 4\left(\frac{1-\delta_1}{5}B_2^6(s) + \frac{1+9(1-\delta_0)(1-\delta_1)}{20}B_3^6(s) + \frac{1-\delta_0}{5}B_4^6(s)\right) \\ &\quad - \left(\frac{3\delta_0^2}{5}B_2^6(s) + \frac{9\delta_0\delta_1}{10}B_3^6(s) + \frac{3\delta_1^2}{5}B_4^6(s)\right) \\ &= \frac{1}{5}(4(1-\delta_1) - 3\delta_0^2)B_2^6(s) + \frac{1}{10}(20 - 18(\delta_0 + \delta_1) + 27\delta_0\delta_1)B_3^6(s) \\ &\quad + \frac{1}{5}(4(1-\delta_0) - 3\delta_1^2)B_4^6(s). \end{aligned}$$

By differentiation rule of Bézier curves $\frac{d}{dt}[\sum_{i=0}^n \mathbf{b}_i B_i^n(t)] = n \sum_{i=0}^{n-1} (\mathbf{b}_{i+1} - \mathbf{b}_i) B_i^{n-1}(t)$, we obtain

$$\begin{aligned} \frac{df(\mathbf{c}(s))}{ds} &= B_1^2(s)[3G_0(\delta_0, \delta_1)B_0^3(s) + G_1(\delta_1, \delta_0)B_1^3(s) \\ &\quad - G_1(\delta_1, \delta_0)B_2^3(s) - 3G_0(\delta_1, \delta_1)B_3^3(s)] \end{aligned} \tag{2.6}$$

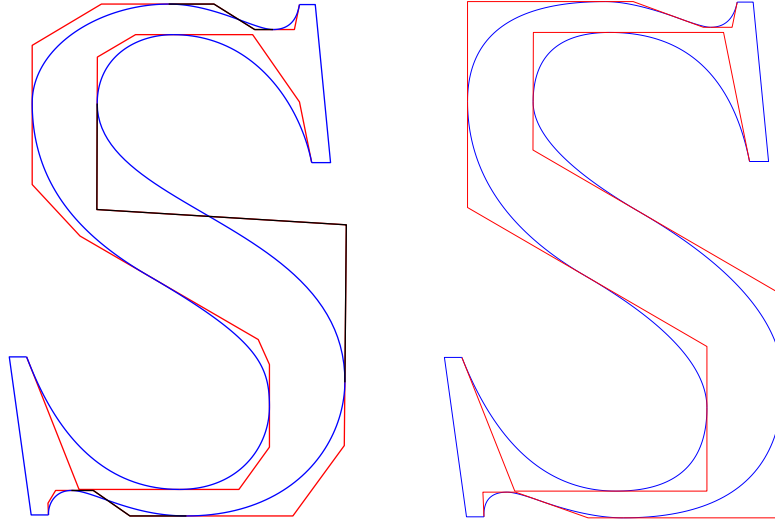


FIGURE 4. Left : The cubic spline curve(blue) consists of 6 line segments and 13 cubic Bézier curves $\mathbf{c}_1(s), \dots, \mathbf{c}_{13}(s)$ with their control polygon(red) in counter-clockwise order. Among them $\mathbf{c}_2(s)$, $\mathbf{c}_9(s)$, and $\mathbf{c}_{11}(s)$ have an inflection point, and their control polygons are drawn by black. Right : The quadratic spline approximation curve(blue) consists of 6 line segments and 16 quadratic Bézier curves with their control polygon(red).

where

$$G_0(u, v) = 4(1 - v) - 3u^2, \quad G_1(u, v) = -6 + 18(1 - u)(1 - v) - 9uv + 8v + 6u^2.$$

The cubic Bézier function in Equation (2.6) can be expressed by poser series form

$$\begin{aligned} & 3[3(\delta_0 + \delta_1) - 4]^2 s^3 - 3[3(\delta_0 + \delta_1) - 4][7\delta_0 + 2\delta_1 - 6]s^2 \\ & + 3[15\delta_0^2 + 9\delta_0\delta_1 - 18\delta_0 + 2\delta_1]s + 3[4(1 - \delta_1) - 3\delta_0^2] \end{aligned}$$

and its zeros in the open interval $(0, 1)$ are the critical points of $f(\mathbf{c}(s))$, since $B_1^2(s)$ cannot have zeros in $(0, 1)$. \square

The cubic equation (2.5) can be solved symbolically by Cardan's solution [22].

3. APPLICATION

In this section we present application of quadratic spline approximation of cubic spline curve to approximate the offset curves of cubic spline. In general the offset curve of planar curve $\mathbf{p}(t) = (x(t), y(t))$, $t \in [a, b]$, cannot easily expressed in polynomial or rational, since the offset curve

$$\mathbf{p}_d(t) = \mathbf{p}(t) + d \frac{(-y'(t), x'(t))}{\sqrt{x'(t)^2 + y'(t)^2}}, \quad t \in [a, b]$$

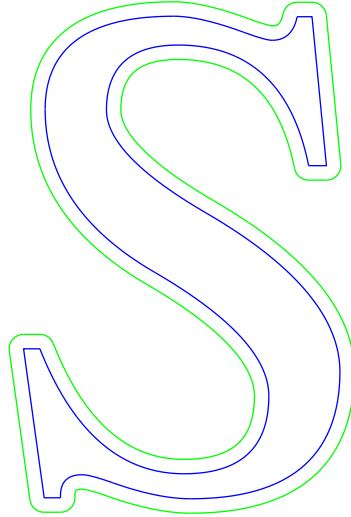


FIGURE 5. The offset curve of the quadratic spline approximation curve(blue) can be exactly expressed by the rational spline curve(green) of degree six which consists of 6 line segments, 8 circular arcs and 16 rational Bézier curves of degree six.

with offset-distance d has radical. But surprisingly the offset curve of any quadratic spline curve can be exactly expressed by rational spline curve of degree six [14, 21]. Thus the rational spline curve of degree six is an approximation of offset curve of cubic spline.

Let the outline of the font 'S' be given by the cubic spline curve as shown in Figure 4. The height of the font is 1. It consists of 13 cubic Bézier curves $\mathbf{c}_i(s)$, ($i = 1, \dots, 13$) and 6 line segments. Each cubic Bézier curve is approximated by quadratic Bézier curve $\mathbf{q}_j(t)$. But the cubic Bézier curves having inflection point, $\mathbf{c}_i(s)$, ($i = 2, 9, 11$), cannot be approximated by one quadratic Bézier curve, so after they are subdivided at the inflection points, quadratic approximations are achieved, $\mathbf{q}_j(t)$, ($j = 2, 3, 10, 11, 13, 14$). Thus the quadratic spline approximation consists of 16 quadratic Bézier curves and 6 line segments, as shown in Figure 4. By Proposition 2.3, we obtain the upper bound of the Hausdorff distance between cubic Bézier curve $\mathbf{c}_i(s)$ and its quadratic approximation curve $\mathbf{q}_j(t)$, as shown in Table 1. Finally, we present the offset curve of the quadratic spline curve for offset distance $d = 0.03$ as shown in Figure 5. It is the rational spline curve of degree six consisting 16 rational Bézier curve of degree six, 6 line segments, and 8 circular arcs.

The maximum of approximation error is 4.15×10^{-2} , which occurs at $\mathbf{c}_5(s)$. If the cubic Bézier curve having the maximum error is subdivided, then the quadratic approximations of two subdivided cubic Bézier curves can be proceed, and the maximum error must be reduced with approximation order 4.

| cubic Bézier curve | quadratic approximation | upper bound of $d_H(\mathbf{c}, \mathbf{q})$ |
|----------------------|--|--|
| $\mathbf{c}_1(s)$ | $\mathbf{q}_1(t)$ | 6.30×10^{-3} |
| $\mathbf{c}_2(s)$ | $\mathbf{q}_2(t), \mathbf{q}_3(t)$ | $5.67 \times 10^{-3}, 6.50 \times 10^{-3}$ |
| $\mathbf{c}_3(s)$ | $\mathbf{q}_4(t)$ | 2.52×10^{-2} |
| $\mathbf{c}_4(s)$ | $\mathbf{q}_5(t)$ | 3.09×10^{-2} |
| $\mathbf{c}_5(s)$ | $\mathbf{q}_6(t)$ | 4.15×10^{-2} |
| $\mathbf{c}_6(s)$ | $\mathbf{q}_7(t)$ | 2.14×10^{-2} |
| $\mathbf{c}_7(s)$ | $\mathbf{q}_8(t)$ | 2.60×10^{-2} |
| $\mathbf{c}_8(s)$ | $\mathbf{q}_9(t)$ | 5.98×10^{-3} |
| $\mathbf{c}_9(s)$ | $\mathbf{q}_{10}(t), \mathbf{q}_{11}(s)$ | $5.37 \times 10^{-3}, 5.60 \times 10^{-3}$ |
| $\mathbf{c}_{10}(s)$ | $\mathbf{q}_{12}(t)$ | 3.59×10^{-2} |
| $\mathbf{c}_{11}(s)$ | $\mathbf{q}_{13}(t), \mathbf{q}_{14}(s)$ | $3.95 \times 10^{-2}, 3.18 \times 10^{-2}$ |
| $\mathbf{c}_{12}(s)$ | $\mathbf{q}_{15}(t)$ | 1.87×10^{-2} |
| $\mathbf{c}_{13}(s)$ | $\mathbf{q}_{16}(t)$ | 2.84×10^{-2} |

TABLE 1. The upper bound $d_F(\mathbf{c}, \mathbf{q})$ of the Hausdorff distance between cubic Bézier curve $\mathbf{c}(s)$ and its quadratic Bézier curve $\mathbf{q}(t)$.

REFERENCES

- [1] Y. J. Ahn. 2001. Conic Approximation of planar curves. *Comp. Aided Desi.* 33(12): 867-872.
- [2] Y. J. Ahn. 2003. Degree reduction of Bézier curves with C^k -continuity using Jacobi polynomials. *Comp. Aided Geom. Desi.* 20: 423-434.
- [3] Y. J. Ahn. 2003. Degree reduction of Bezier curves using constrained Chebyshev polynomials of second kind. *ANZIAM J.* 45: 195-205.
- [4] Y. J. Ahn. 2005. Helix approximation with conic and quadratic Bézier curves. *Comp. Aided Geom. Desi.* 22(6): 551-565.
- [5] Y. J. Ahn, H. O. Kim. 1997. Approximation of circular arcs by Bézier curves. *J. Comp. Appl. Math.* 81: 145-163.
- [6] Y. J. Ahn, H. O. Kim. 1998. Curvatures of the quadratic rational Bézier curves. *Comp. Math. Appl.* 36(9):71-83.
- [7] Y. J. Ahn, H. O. Kim, K. Y. Lee. 1998. G^1 arc spline approximation of quadratic Bézier curves. *Comp. Aided Desi.* 30(8):615-620.
- [8] Y.J. Ahn, Y. S. Kim, Y. S. Shin. 2004. Approximation of circular arcs and offset curves by Bezier curves of high degree. *J. Comp. Appl. Math.* 167, 181-191.
- [9] B. Bastl, B. Juttler, J. Kosinka, M. Lávcka. 2008. Computing exact rational offsets of quadratic triangular Bezier surface patches *Comp. Aided Desi.* 40: 197-209.
- [10] M. G. Cox, P. M. Harris. 1990. The approximation of a composite Bézier cubic curve by a composite Bézier quadratic curve. *IMA J. Nume. Anal.* 11:159-180.
- [11] E. F. Eisele. 1994. Chebychev approximation of plane curves by splines. *J. Appr. Theo.* 76:133-148.
- [12] G. Elber, I.K. Lee, M.S. Kim. 1997. Comparing offset curve approximation method. *IEEE comp. grap. appl.* 17(3): 62-71.
- [13] G. Farin. *Curves and Surfaces for Computer Aided Geometric Design.* Morgan-Kaufmann, San Francisco, 2002.
- [14] R. T. Farouki, T. W. Sederberg. 1995. Analysis of the offset to a parabola. *Comp. Aided Geom. Desi.* 12(6): 639-645.

- [15] R. T. Farouki, J. Hass. 2007. Evaluating the boundary and covering degree of planar Minkowski sums and other geometrical convolutions. *J. Comp. Appl. Math.* 209: 246-266.
- [16] M. Floater. 1995. High-order approximation of conic sections by quadratic splines. *Comp. Aided Geom. Desi.* 12(6): 617-637.
- [17] M. Floater. 1997. An $\mathcal{O}(h^{2n})$ Hermite approximation for conic sections. *Comp. Aided Geom. Desi.* 14:135–151.
- [18] T. F. Hain, A. L. Ahmad, S. Venkat, R. Racherla, D. D. Langan. 2005. Fast, precise flattening of cubic Bezier path and offset curves. *Comp. Grap.* 29(5): 656–666.
- [19] S. H. Kim and Y. J. Ahn. 2007. Approximation of circular arcs by quartic Bezier curves. *Comp. Aided Desi.* 39(6): 490-493.
- [20] I.K. Lee, M.S. Kim, and G. Elber. 1996. Planar curve offset based on circle approximation. *Comp. Aided Desi.* 28:617–630.
- [21] W. Lü. 1992. Rational offsets by reparametrizations. preprint.
- [22] R. W. D. Nickalls. 1993. A new approach to solving the cubic: Cardan’s solution revealed. *Math. Gaze.* 77: 354–359.
- [23] M. Peternell. 2009. Rational two-parameter families of spheres and rational offset surfaces. *J. Symb. Comp.* to appear.
- [24] J. Sanchez-Reyes. 2007. Offset-rational sinusoidal spirals in Bezier form. *Comp. Aided Geom. Desi.* 24(3): 142-150.
- [25] H. Y. Zhao, G. J. Wang. 2007. Error analysis of reparametrization based approaches for curve offsetting. *Comp. Aided Desi.* 39: 142-148.