

EFFICIENT PARAMETERS OF DECOUPLED DUAL SINGULAR FUNCTION METHOD

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ABSTRACT. The solution of the interface problem or Poisson problem with concave corner has singular perturbation at the interface corners or singular corners. The decoupled dual singular function method (DDSFM) which exploits the singular representations of the solutions was suggested in [3, 9] and estimated optimal accuracy in [10]. The convergence rates consist with theoretical results even for the problems with very strong singularity, with the efficiency depending on parameters used in the methods. Furthermore the errors in L^2 and L^∞ -spaces display some oscillation, in the cases with meshsize not small enough. In this paper, we present an answer to remove the oscillation via numerical experiments. We observe the effects of parameters in DDSFM, and show the consisting efficiency of the method over the strong singularity.

1. INTRODUCTION

Let Ω be an open, bounded polygonal domain in \mathcal{R}^2 . Consider the following model Laplace problem (called interface problem)

$$\begin{cases} -\nabla \cdot (a\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma := \partial\Omega, \end{cases} \quad (1.1)$$

where the diffusion coefficient $a(x)$ is a given piecewise constant function on subdomains Ω_j , which are also assumed to be polygonal, and f is a given function in $L^2(\Omega)$. This interface problem is simply reduced to the Poisson problem if $a(x)$ is constant on Ω . For the interface problems and the Poisson problems on non-convex domains, the precise singular function representations for solutions are well-known (see [1, 6, 7, 8, 11, 13, 14]). In [3, 5] the authors

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derived a new extraction formula for the stress intensity factor in term of integration of the regular part of the solution, and used it together with the singular representation, to pose a variational problem for the regular part of the solution. The optimal regularity of the regular part gives the h^1 -error and h^2 -error in computational results, in H^1 - seminorm and L^2 -norm, respectively, although the optimal theoretical error analysis in L^2 -norm is obtained recently in [10]. This approach for the interface problem is considered in [9], where two examples of interface problems were given with numerical computations. Although the results shows the method works with high accuracy, it shows a little oscillating convergence rate. We recall that we have used a cut-off function using a polynomial degree 5 to give the singular function representations which plays an important role in the method.

In this paper we find that the method works well without oscillating in convergence rate if we use polynomials with degree 7 and study the effects of the choice of parameters ρ and R in the definition of the cut-off function. We also give several examples of interface problems with singular points, in the interior or on the boundary, to show the method works with a stable results for a series of examples with dramatically increasing singularities.

Here we note that the adaptive mesh refinement has strong points for the treatment of the singular problem without knowledge of the singularities. One of the issue is to find a posteriori error estimates which is robust. But for some special configurations of diffusion coefficients it has been shown that there are no robust interpolation operators, for example checkerboard distribution of coefficients [15, 16].

The first example of this paper is about interface problems with checkerboard distribution of coefficients, having a strong singularity. Although we do not have the theoretical proof yet, the computational results indicate that the method is an accurate numerical solver with scale of the errors being independent of the jumps of the diffusion coefficients,

We will use the standard notation and definitions for the Sobolev spaces $H^t(\Omega)$ for $t \geq 0$; the standard associated inner products are denoted by $(\cdot, \cdot)_{t,\Omega}$, and their respective norms and seminorms are denoted by $\|\cdot\|_{t,\Omega}$ and $|\cdot|_{t,\Omega}$. The space $L^2(\Omega)$ is interpreted as $H^0(\Omega)$, in which case the inner product and norm will be denoted by $(\cdot, \cdot)_\Omega$ and $\|\cdot\|_\Omega$, respectively, although we will omit Ω if there is no chance of misunderstanding. $H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma\}$.

2. REVIEW OF ALGORITHM AND ERROR ANALYSIS

The cut-off function plays an important role in isolating the singular behavior of the problem. We start this section with the definition of cut-off functions, which involves two parameters to be analyzed in the next section. For this end, set

$$B(r_1; r_2) = \{(r, \theta) : r_1 < r < r_2 \text{ and } 0 < \theta < \omega\} \cap \Omega \quad \text{and} \quad B(r_1) = B(0; r_1).$$

2.1. Cut-off functions. We define two families of cut-off functions of r as follows :

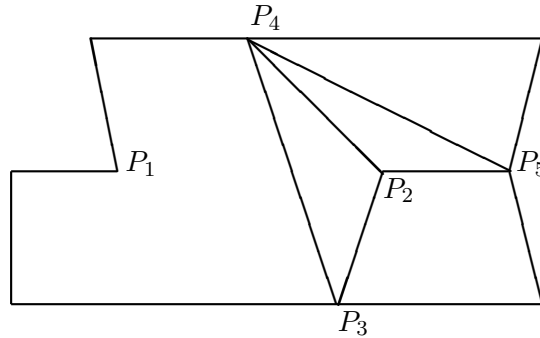
$$\eta_{5,\rho}(r) = \begin{cases} 1 & \text{in } B(\frac{1}{2}\rho R), \\ \frac{15}{16} \left\{ \frac{8}{15} - p(r) + \frac{2}{3}p(r)^3 - \frac{1}{5}p(r)^5 \right\} & \text{in } \bar{B}(\frac{1}{2}\rho R; \rho R), \\ 0 & \text{in } \Omega \setminus \bar{B}(\rho R), \end{cases} \quad (2.1)$$

and

$$\eta_{7,\rho}(r) = \begin{cases} 1 & \text{in } B(\frac{1}{2}\rho R), \\ \frac{1}{32} \{16 - 35p(r) + 35p(r)^3 - 21p(r)^5 + 5p(r)^7\} & \text{in } \bar{B}(\frac{1}{2}\rho R; \rho R), \\ 0 & \text{in } \Omega \setminus \bar{B}(\rho R), \end{cases} \quad (2.2)$$

where $p(r) = \frac{4r}{\rho R} - 3$. Here, ρ is a parameter in $(0, 2]$ and $R \in \mathcal{R}$ is a fixed number which will be determined so that the singular part $\eta_2 s$ has the same boundary condition as the solution u of the model problems.

Note $\eta_{5,\rho}(r)$ and $\eta_{7,\rho}(r)$ are C^2 and C^3 , respectively. We will use the same notation, η or η_ρ , etc., for these cut-off functions from now on until section 4, where the computational results of several examples using these two different choices of cut-off functions are compared.



(a) corner singular point : P_1 (b) interior interface singular point : P_2
 (c) boundary interface singular point : P_3, P_4, P_5

FIGURE 1. Examples of singular points: corner, interior interface, and boundary interface singularities

2.2. Singularities and variational problem. First we need to identify the singular points of the problem and singular functions. As depicted in Figure 1, there are three types of singular points; corner singularities for the Poisson problem on the polygonal or corner with constant diffusion coefficient (see (a)), the interior and boundary singularities for the interface problem (see (b) and (c)).

The precise form of the singular function and dual singular function can be found in many literature. For the corner singularity of Poisson problem we refer [3, 4], for example. In the case of the interface problem we need to solve a Sturm-Liouville problem to get singularities and the corresponding eigenfunctions, then the singular functions and dual singular functions as in [9].

Moreover we assume there is only one singular point, since the complexity caused by the multiple singular points can be easily treated by the help of Sherman-Morrison formula.

• Singular function representation

Using a cut-off function above, the solution of (1.1) has the following singular function representation

$$u = w + \sum_{l=1}^L \kappa_l \eta_\rho s_l, \tag{2.3}$$

with $w \in H^2(\Omega_j)$, $1 \leq j \leq L$, where the subdomain Ω_j of Ω is a polygonal subset of Ω and the diffusion coefficient is constant there, and κ_l is the so-called stress intensity factors for $1 \leq l \leq L$. Note we call w the regular part of the solution and it depends on the choice of ρ in (2.3) but the method works well for any $0 < \rho \leq 1$. Therefore, we may assume $\rho = 1$ so that w is fixed for the simplicity. The complexity of linear system due to the multiplicity of singular points and/or singular functions can be efficiently treated by Sherman-Morrison formula.

• Extraction formula and variational problem

Lemma 2.1. *The stress intensity factors κ_l for $1 \leq l \leq L$ for the interface problem can be expressed in terms of w by the following extraction formula*

$$\kappa_l = \frac{1}{2\alpha_l} [(f, \eta_2 s_{-l}) + (aw, \Delta(\eta_2 s_{-l}))]. \tag{2.4}$$

Now we derives a variational problem for the regular part of the solution, whose well-posedness is proved in [9, 3]. The singular function representation (2.3) of u and Lemma 2.1 yield the variational problem for the regular part of the solution; to find $w \in H_0^1(\Omega)$ such that

$$a(w, v) = g(v) \quad \forall v \in H_0^1(\Omega), \tag{2.5}$$

where the bilinear and linear forms are defined by

$$a(w, v) = (a\nabla w, \nabla v) - \sum_{l=1}^L \frac{1}{2\alpha_l} (aw, \Delta(\eta_2 s_{-l})) (a \Delta(\eta_\rho s_l), v) \tag{2.6}$$

and

$$g(v) = (f, v) + \sum_{l=1}^L \frac{1}{2\alpha_l} (f, \eta_2 s_{-l}) (a \Delta(\eta_\rho s_l), v), \tag{2.7}$$

respectively.

Here we note that $s_l \in H^{1+\alpha_l-\epsilon}(\Omega_{m_i})$ and $s_{-l} \in H^{1-\alpha_l-\epsilon}(\Omega_{m_i})$ for any $\epsilon > 0$. Moreover we see the integrals in both (2.6) and (2.7) are well-defined since the cutoff functions η_2 and η_ρ are 1 around the vertex p and $\Delta(\eta_2 s_{-l}) = \Delta(s_{-l}) = 0$ and $\Delta(\eta_\rho s_l) = \Delta(s_l) = 0$ there.

We refer [3, 9] for the existence and uniqueness of the solution w of the problem (2.5). Once the solution is obtained, the stress intensity factors κ_l and the solution u of (1.1) are obtained by (2.4) and (2.3).

2.3. Summary of the algorithm and error analysis. In this subsection we list the error analysis of the method in the standard norms, $\|\cdot\|_1$ and $\|\cdot\|$, carried out with a regular triangulation and continuous piecewise linear finite element space V_h (see [3, 9, 10]).

The finite element approximation to w is to seek $w_h \in V_h$ such that

$$a(w_h, v) = g(v) \quad \forall v \in V_h, \quad (2.8)$$

with error bounds

$$\|w - w_h\|_1 \leq Ch\|f\| \quad \text{and} \quad \|w - w_h\| \leq Ch^2\|f\|. \quad (2.9)$$

Now approximations $\kappa_{l,h}$ and u_h to the stress intensity factors κ_l and the solution u of (1.1) can be computed according to (2.4) and (2.3) as follows

$$\kappa_{l,h} = \frac{1}{2\alpha_l} (aw_h, \Delta(\eta_2 s_{-l}))_{B(R;2R)} + \frac{1}{2\alpha_l} (f, \eta_2 s_{-l})_{B(2R)} \quad (2.10)$$

and

$$u_h = w_h + \sum_l \kappa_{l,h} \eta_\rho s_l, \quad (2.11)$$

respectively, with error bounds

$$|\kappa_l - \kappa_{l,h}| \leq Ch^2\|f\|, \quad \|u - u_h\|_1 \leq Ch\|f\| \quad \text{and} \quad \|u - u_h\| \leq Ch^2\|f\|. \quad (2.12)$$

Finally we summarize the procedure we get the approximation u_h from with solution w_h and $\kappa_{l,h}$, which we might call Decoupled Dual Singular Function Methods (DDSFM) as follows;

Algorithm 1: (Decoupled Dual Singular Function Methods)

- Find the finite element approximation $w_h \in V_h$ such that

$$a(w_h, v) = g(v) \quad \forall v \in V_h. \quad (2.13)$$

- Compute $\kappa_{l,h}$ using (2.10) for $1 \leq l \leq L$.
- Then we have the solution u_h by (2.11).

3. PARAMETERS ρ AND R IN THE ALGORITHM

In this section we examine the effects of two parameters ρ and R in the cut-off functions to the convergence of the Algorithm. If we consider the usual variational form, the speed of convergence is roughly in inverse proportion to the ratio of two constants appearing in the continuity and coercivity inequalities for the bilinear form due to the Céa's Theorem. Note the variational problem (2.5) does not satisfy the usual coercivity and may not use the Céa's Theorem.

Nevertheless we observe that only the second terms in the bilinear form (2.6) contain the parameters ρ and R . So we compute the norms of Laplacians of the singular and dual singular function multiplied by the cut-off functions delicately, then see the dependence of the continuity constant to the parameters ρ and R in the algorithm.

Lemma 3.1.

$$\Delta(\eta_\rho s_l) = (\partial_{rr}\eta_\rho + \frac{1}{r}(1 + \frac{2\pi}{\omega})\partial_r\eta_\rho)s_l := p_1(r)r^{\alpha_l}\Theta_l(\theta) \tag{3.1}$$

on $B(\frac{\rho R}{2}; \rho R)$ and

$$\Delta(\eta_2 s_{-l}) = (\partial_{rr}\eta_2 + \frac{1}{r}(1 - \frac{2\pi}{\omega})\partial_r\eta_2)s_{-l} := p_2(r)r^{-\alpha_l}\Theta_l(\theta), \tag{3.2}$$

on $B(R; 2R)$ where p_i are functions of r , only dependent on the cut-off functions.

Proof. Use the facts $\Delta s_l = 0$ and that

$$\Delta(\eta_\rho s_l) = \Delta(\eta_\rho)s_l + 2\nabla\eta_\rho \cdot \nabla s_l = (\partial_{rr}\eta_\rho + \frac{1}{r}\partial_r\eta_\rho)s_l + 2\eta_\rho(\cos\theta, \sin\theta)^T \cdot \nabla s_l.$$

A similar method can be applied for the dual function s_{-l} . ■

Lemma 3.2. For any $0 < \rho \leq 1$, we have that

$$\|a^{\frac{1}{2}}\Delta(\eta_2 s_{-l})\|_{B(R; 2R)} \leq C_4 R^{-\alpha_l - 1} \tag{3.3}$$

and

$$\|a^{\frac{1}{2}}\Delta(\eta_\rho s_l)\|_{B(\frac{\rho R}{2}; \rho R)} \leq C_6(\rho R)^{\alpha_l - 1}. \tag{3.4}$$

For the observation for the effect of parameters on the coefficient in the error analysis we have the following lemma;

Lemma 3.3. For $0 < \rho \leq 1$, there exist positive constants C_l such that

$$|(a\phi, \Delta(\eta_2 s_{-l})) (a\Delta(\eta_\rho s_l), \psi)| \leq \frac{C_4 C_6 \rho^{\alpha_l - 1}}{R^2} \|a^{\frac{1}{2}}\phi\| \|a^{\frac{1}{2}}\psi\|. \tag{3.5}$$

Proof. The lemma is immediate by the Lemma 3.2. ■

Now we have the following observations regarding the effects of the parameters R and ρ to the algorithm;

- O1::** The coefficient on the right hand side of (3.5) has negative powers -2 and $\alpha_l - 1$ on R and ρ , respectively.
- O2::** Since $0 < \alpha_l < 1$, we need to choose the parameter $\rho = 1$ for a smaller continuity coefficient.
- O3::** The bigger the number R , the less the coefficient on the right hand-side in Lemma 3.3. So, we need to choose bigger R such that $B(0; 2R) \subset \Omega$ does not contain any other singular points or corner points.

4. NUMERICAL EXPERIMENTS

The purpose of this section is to present various numerical tests, which complete the purpose of the paper. Since we present numerical results for singularity at a corner in [9], we carry out numerical test on a square domain $\Omega = (-1, 1)^2$ in Figure 2 with interior singularity at origin.

The purposes of Example 1 are to assert that

P1:: the method works independently to the jump of the diffusion of the interface problem.

P2:: It works well even for the interface problem with strong singular solution in H^α ($\alpha \approx 1.0550927$).

The purposes of Example 2 are to assert that

P3:: We can remove the oscillation in the rate of convergence error by using the polynomial degree 7 instead of degree 5 in the cut-off function.

P4:: We can reduce the error by using bigger R rather than smaller one.

Here, we use the following notations for the H^1 -seminorm and relative H^1 -seminorm;

$$\|E\|_{H^1} = \|\nabla(u - u_h)\| \quad \text{and} \quad \|E\|_R = \frac{\|a^{\frac{1}{2}}\nabla(u - u_h)\|}{\|a^{\frac{1}{2}}\nabla u\|}.$$

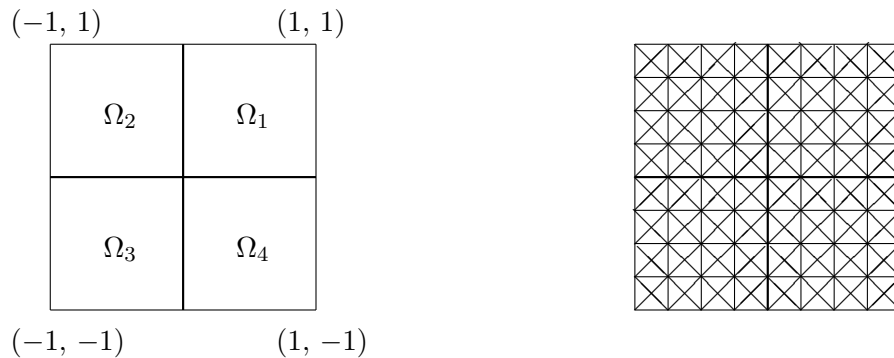


FIGURE 2. A square partitioned into four squares.

Our first example is an interface problem with an interior singular point. The results show that our methods is independent of the jump of the diffusion coefficients.

4.1. Example 1 : Comparison numerical results for the variation of diffusion coefficients.

We consider the interface problem with an interior singular interface point at the origin;

$$\begin{cases} -\nabla \cdot (a \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.1)$$

where $\Omega = (-1, 1)^2$ with four square subdomains as in Figure 2. The diffusion coefficients a_i in each subdomain Ω_i are defined as follows;

Case 1: $a_1 = a_3 = 1$, $a_2 = 25$, and $a_4 = 50$.

Case 2: $a_1 = a_3 = 1$, $a_2 = 100$, and $a_4 = 200$.

Case 3: $a_1 = a_3 = 1$, $a_2 = 400$, and $a_4 = 800$.

Note the coefficients a_2 and a_4 , on Ω_2 and Ω_4 , are chosen to jump 4 times, respectively. For each case we can compute the positive eigenvalues λ and the corresponding eigenfunctions of the Sturm-Liouville problem, then determine the singular functions s . The following are the square roots $\sqrt{\lambda}$ of the computed positive eigenvalues which determine the singularities (power to r) for each case;

Case 1: $\alpha \approx 0.21801835634053335$,

Case 2: $\alpha \approx 0.1099460764271882$,

Case 3: $\alpha \approx 0.05509274836764086$.

We note that the more the coefficients jump, the smaller the power is and the singularity of the solution increases. Now we introduce the forcing term f in the problem (4.1);

$$f = -a_i \left(-\frac{6}{a_i} x(y^2 - y^4) + \frac{1}{a_i} (x - x^3)(2 - 12y^2) + \Delta(\eta_2 s) \right) \quad \text{on } \Omega_i,$$

so that the exact solution has the singular decomposition;

$$u = w_\rho + \eta_\rho s,$$

where η_ρ is the cut-off function with $R = 1/2$ and $\rho = 1$ in (2.1). Thus, $w := w_\rho$ is the regular part of the solution having the form of

$$w = \frac{1}{a_i} (x - x^3)(y^2 - y^4) + (\eta_2 - \eta_\rho) s \quad \text{on } \Omega_i, \quad i = 1, 2, 3, 4.$$

The computational results for three cases are given in Table 1, 2, and 3 and there are only small difference among the results. So we can conclude that DDSFM works robustly for very big jump coefficients with strong singularities. We remark that the oscillation of errors in $L^2(\Omega)$ and $L^\infty(\Omega)$ -spaces due to insufficient of regularity of cut off function (2.1). We will compare numerical behaviors between cut off functions (2.1) and (2.2) in §4.2.

4.2. Example 2 : Two choices of the parameter R and the polynomial with degree 5 and 7 in the cut-off function.

The domain and subdomains are the same as in Example 1 and $a_1 = a_3 = R_0$ and $a_2 = a_4 = 1$, where R_0 can be chosen later so that we have particular singularity $\alpha = 0.1$ (see [2, 12]). In fact, $R_0 \approx 161.4476387975881$.

h	1/16	1/32	1/64	1/128	1/256	1/512
$\ E\ _{L^2}$	0.121169	0.00641669	0.00703718	0.000803131	0.000337706	6.96411e-05
	Order	4.239048	-0.133168	3.131290	1.249868	2.277757
$\ E\ _{L^\infty}$	0.312583	0.016195	0.0183967	0.00223744	0.000905891	0.00018592
	Order	4.270619	-0.183899	3.039526	1.304440	2.284656
$\ E\ _{H^1}$	1.23423	0.45426	0.231951	0.113141	0.0565359	0.0282413
	Order	1.442021	0.969698	1.035698	1.000883	1.001361
$\ E\ _R$	0.117836	0.0291697	0.0153021	0.00705695	0.00351798	0.00175353
	Order	2.014238	0.930741	1.116613	1.004298	1.004485
κ_h	1.1937	1.01893	1.01121	1.00177	1.00059	1.00013
	Order	3.355078	0.755888	2.662965	1.584962	2.182203

TABLE 1. Error decay for case 1 of §4.1

h	1/16	1/32	1/64	1/128	1/256	1/512
$\ E\ _{L^2}$	0.189589	0.00979614	0.0117825	0.00144784	0.000619979	0.000133304
	Order	4.274518	-0.266360	3.024672	1.223611	2.217499
$\ E\ _{L^\infty}$	0.515027	0.0263781	0.0314316	0.00398715	0.00165554	0.00035362
	Order	4.287235	-0.252875	2.978786	1.268056	2.227030
$\ E\ _{H^1}$	1.80414	0.501356	0.266292	0.124206	0.0619946	0.0308978
	Order	1.847404	0.912826	1.100274	1.002520	1.004638
$\ E\ _R$	0.188638	0.0308873	0.0180715	0.00728764	0.00361533	0.00178846
	Order	2.610535	0.773297	1.310193	1.011324	1.015410
κ_h	1.31658	1.02625	1.0187	1.00281	1.001	1.00022
	Order	3.592181	0.489279	2.734396	1.490570	2.184425

TABLE 2. Error decay for case 2 of §4.1

The exact solution of the interface problem with a vanishing right-hand side f can be found in [2] or computed by using the formulas derived by R. B. Kellogg([8]). To use our approach, we split the exact solution u to $u = w + \lambda\eta_\rho s$, where η_ρ is the same cut-off function as in Example 1. So, we have $\alpha = 0.1$ and can check the singular function $s = r^\alpha(C_i \cos(\alpha\theta) + D_i \sin(\alpha\theta))$ have the coefficients C_i, D_i on each subdomains Ω_i (See Figure 2) as follows;

$$\begin{aligned}
C_1 &= 1.0, & D_1 &= 0.07870170682462, \\
C_2 &= 2.97537668119027, & D_2 &= -12.39333580609424, \\
C_3 &= -0.92673635140003, & D_3 &= -0.38386676549405, \\
C_4 &= -6.65950276215729, & D_4 &= 10.86732081786520.
\end{aligned} \tag{4.2}$$

We first remark that the convergence rates are independent on R , but the errors for the bigger $R = 1/2$ in Tables 4 and 6 are much smaller than those for the small $R = 1/8$ in Tables 5

h	1/16	1/32	1/64	1/128	1/256	1/512
$\ E\ _{L^2}$	0.248176	0.0128642	0.0158906	0.00204376	0.000888561	0.000197039
	Order	4.269930	-0.304812	2.958876	1.201683	2.172990
$\ E\ _{L^\infty}$	0.685167	0.0343631	0.0425568	0.00554794	0.00235945	0.000518869
	Order	4.317524	-0.308530	2.939366	1.233501	2.185008
$ E _{H^1}$	2.34664	0.529674	0.294471	0.130531	0.0650709	0.0323253
	Order	2.147420	0.846979	1.173733	1.004308	1.009349
$ E _R$	0.255358	0.0325948	0.0212676	0.00750582	0.00370203	0.00181015
	Order	2.969808	0.615985	1.502576	1.019693	1.032207
κ_h	1.42513	1.03203	1.02544	1.00376	1.00141	1.00032
	Order	3.730408	0.332325	2.758294	1.415037	2.139551

TABLE 3. Error decay for case 3 of §4.1

h	1/16	1/32	1/64	1/128	1/256	1/512
$\ E\ _{L^2}$	0.00212664	0.000302188	0.000111722	1.8242e-05	5.05712e-06	1.13813e-06
	Order	2.815057	1.435533	2.614577	1.850876	2.151651
$\ E\ _{L^\infty}$	0.00414278	0.00119037	0.000226678	3.70706e-05	1.04631e-05	2.17368e-06
	Order	1.799189	2.392694	2.612297	1.824965	2.267099
$ E _{H^1}$	0.0463328	0.0216787	0.0107002	0.00532138	0.00266172	0.0013307
	Order	1.095756	1.018640	1.007765	0.999442	1.000173
$ E _R$	0.0388715	0.0159448	0.00766791	0.00376714	0.001879	0.000938677
	Order	1.285627	1.056181	1.025364	1.003505	1.001264

TABLE 4. Error decay with $R = 1/2$ and cut-off function $\eta = \eta_{5,1}$ in §4.2

h	1/16	1/32	1/64	1/128	1/256	1/512
$\ E\ _{L^2}$	0.0257975	0.0027329	0.00138712	0.000150095	7.75654e-05	4.84607e-06
	Order	3.238727	0.978340	3.208145	0.952391	4.000526
$\ E\ _{L^\infty}$	0.0578551	0.0111715	0.00484416	0.000779646	0.000270837	3.98846e-05
	Order	2.372621	1.205505	2.635355	1.525394	2.763521
$ E _{H^1}$	0.396396	0.0815404	0.0466302	0.018517	0.00938305	0.00464047
	Order	2.281356	0.806250	1.332414	0.980722	1.015786
$ E _R$	0.352453	0.0801992	0.0380502	0.0137496	0.00671486	0.00328066
	Order	2.135771	1.075684	1.468514	1.033960	1.033371

TABLE 5. Error decay with $R = 1/8$ and cut-off function $\eta = \eta_{5,1}$ in §4.2

h	1/16	1/32	1/64	1/128	1/256	1/512
$\ E\ _{L^2}$	0.00115259	0.000315032	8.21328e-05	2.06608e-05	5.17576e-06	1.29526e-06
	Order	1.871309	1.939468	1.991062	1.997053	1.998529
$\ E\ _{L^\infty}$	0.00220923	0.000671963	0.000177266	4.42892e-05	1.11713e-05	2.79298e-06
	Order	1.717090	1.922466	2.000889	1.987158	1.999920
$\ E\ _{H^1}$	0.0494193	0.0255158	0.0128035	0.00639884	0.00319865	0.00159922
	Order	0.953684	0.994853	1.000656	1.000347	1.000095
$\ E\ _R$	0.0402564	0.0191564	0.00924836	0.00456849	0.00227672	0.00113741
	Order	1.071392	1.050557	1.017480	1.004761	1.001204

TABLE 6. Error decay with $R = 1/2$ and cut-off function $\eta = \eta_{7,1}$ in §4.2

h	1/16	1/32	1/64	1/128	1/256	1/512
$\ E\ _{L^2}$	0.244612	0.00570191	0.000336002	8.84486e-05	1.81476e-05	4.50673e-06
	Order	5.422906	4.084903	1.925559	2.285060	2.009626
$\ E\ _{L^\infty}$	0.554409	0.0176703	0.00271292	0.000697018	0.000161296	3.91166e-05
	Order	4.971552	2.703408	1.960579	2.111485	2.043858
$\ E\ _{H^1}$	1.92219	0.0928789	0.0446906	0.0223462	0.0111635	0.00557634
	Order	4.371256	1.055379	0.999942	1.001240	1.001399
$\ E\ _R$	2.13053	0.0848877	0.0365569	0.0167652	0.00805581	0.00397733
	Order	4.649513	1.215412	1.124674	1.057368	1.018229

TABLE 7. Error decay with $R = 1/8$ and cut-off function $\eta = \eta_{7,1}$ in §4.2

and 7. In addition, we can observe oscillation of convergence order in Tables 4 and 5 for the case using $\eta = \eta_{5,1}$ in (2.1). One answer to remove the oscillation is to compute on very fine mesh, but it is not best answer. We observe this perturbation is due to insufficiency of regularity of cut off function η and carry out same experiment with $\eta = \eta_{7,1}$ in (2.2) and get a positive answer to remove the perturbations in Tables 6 and 7 with $\eta = \eta_{7,1}$.

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