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A QUATNARY APPROXIMATING 4-POINT SUBDIVISION SCHEME

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ABSTRACT. In this work, we introduce a new quatnary approximating subdivision scheme for curve and deal with its analysis (convergence and regularity) using Laurent polynomials method. We also discuss various properties, such as approximation order and support of basic limit function.

1. INTRODUCTION

Subdivision scheme ([1, 2, 3, 4, 5]) is a very efficient method to generate smooth curves or surfaces from a set of control points or nets through iterative refinements. It plays a significant role in Computer Aided Geometry Design and wavelets analysis. Its popularity is due to the facts that subdivision algorithms are simple and suitable for computer applications, easy to implement.

Each subdivision scheme is associated with a mask $\mathbf{a} = \{a_i \in \mathbb{R} : i \in \mathbb{Z}^s\}$, where $s = 1$ in the curve case and $s = 2$ in the surface case. The (stationary) subdivision scheme is a process which recursively defines a sequence of control points $f^k = \{f_i^k : i \in \mathbb{Z}^s\}$ by a rule of the form with a mask $\mathbf{a} = \{a_i : i \in \mathbb{Z}^s\}$

$$f_i^{k+1} = \sum_{j \in \mathbb{Z}^s} a_{i-Mj} f_j^k, \quad k \in \{0, 1, 2, \dots\}$$

which is denoted formally by $f^{k+1} = Sf^k$. Then a point of f^{k+1} is defined by a finite affine combination of points in f^k. Here M is an $s \times s$ integer matrix such that $\lim_{n\to\infty} M^{-n} = 0$. The matrix M is called a dilation matrix. Binary (or dyadic), ternary ([6, 7]) and quatnary subdivision methods are schemes with the matrices $M = 2I, M = 3I$ and $M = 4I$, respectively, for the $s \times s$ identity matrix I.

Starting with given control points $f^0 = \{f_i^0 = (i, f_i^0) \in \mathbb{R}^2 : i \in \mathbb{Z}\}$, the stationary quatnary subdivision scheme for curve is a process which recursively defines a sequence of

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control points $f^{k+1} = \{f_i^{k+1} = (x_i^{k+1}, f_i^{k+1}) \in \mathbb{R}^2 : i \in \mathbb{Z}\}$ by a finite linear combination of control points f^k with mask $\mathbf{a} = \{a_i \in \mathbb{R} : i \in \mathbb{Z}\}$;

$$f_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-4j} f_j^k, \quad k = 0, 1, 2, \ldots.$$

A point of f^{k+1} is defined by a finite linear combination of points in f^k with four different rules. If a subdivision scheme retains the point of level k as a subset of point of level $k+1$, it is called an interpolating scheme. Otherwise, it is termed approximating. Throughout the work, we consider quatnary approximating scheme with a mask of finite support for curves.

Ko et al. [8] obtained the mask of binary $2n$-point interpolating schemes, ternary 4-point interpolating scheme using symmetry and necessary condition for the smoothness. Ko et al. [10] presented explicitly a general formula for the mask of (2n+4)-point symmetric subdivision scheme with two parameters that reproduce of degree $\leq 2n+1$. The proposed scheme is a generalization of the Deslauriers and Dubuc scheme that includes various other well-known subdivision schemes by varying the values of two parameters. It is well-known that a higher approximation order does not guarantee a higher regularity. The creation of highly smooth curves or surfaces via a subdivision scheme and the shortness of the support size of its mask are two mutually conflicting requirements. That is, the increase of the smoothness of a subdivision scheme results in that of the support size, which leads to an increase in computational effort. Our objective is to find a quatnary approximating scheme derived from cubic polynomial interpolation, which has smaller support, comparing to binary 4 point and ternary 4 point scheme. Here is an outline of this work. In Section 2, we derive the construction of quatnary 4-point approximating subdivision scheme using the polynomial reproducing property. We establish in Section 3 a basic theory for subdivision schemes and analyze the proposed scheme. Finally, in Section 4, we discuss the approximation order and the support of basic limit function of the proposed scheme and compare our results with other well-known binary and ternary schemes.

2. Construction of quatnary 4-point approximating subdivision scheme

In this section, we construct a new quatnary 4-point approximating scheme, which reproduce polynomials up to degree three. The space of all polynomials of degree $\leq n$ will be denoted by π_n. We shall say that a subdivision scheme S has the mth degree polynomial reproducing property (PRP) if it holds that

$$\sum_{k\in\mathbb{Z}} a_{j-2k} p(k) = p\left(\frac{j}{2}\right), \quad j \in \mathbb{Z}, \quad p \in \pi_m.$$

The Theorem 4.2 connects the polynomial reproducing property with the approximation order of the scheme. Using polynomial reproducing property, we can easily obtain the approximation order. That is why we use polynomial reproducing property to obtain the mask of the proposed scheme.

To obtain a quatnary 4-point approximating subdivision scheme, we derive the mask of this scheme by evaluation at $1/8, 3/8, 5/8$ and $7/8$ on local cubic interpolation.

Let $\{L_i(x)\}_{i=-1}^{2}$ be the fundamental Lagrange polynomials to the node points $\{-1, 0, 1, 2\}$ given by

$$L_{-1}(x) = -\frac{x(x-1)(x-2)}{6}, \quad L_0(x) = \frac{(x+1)(x-1)(x-2)}{2},$$

and

$$L_1(x) = -\frac{x(x+1)(x-2)}{2}, \quad L_2(x) = \frac{x(x+1)(x-1)}{6}.$$

We sample the data $(j, f_j), j = i-1, i, i+1, i+2$ from an arbitrarily given cubic polynomial p_3;

$$p_3(j) = f_j, \quad j = i-1, i, i+1, i+2,$$

and request

$$f_{4i}^1 = p_3\left(i + \frac{1}{8}\right), f_{4i+1}^1 = p_3\left(i + \frac{3}{8}\right), f_{4i+2}^1 = p_3\left(i + \frac{5}{8}\right), f_{4i+3}^1 = p_3\left(i + \frac{7}{8}\right).$$

Since our scheme is stationary, uniform and the space of polynomials up to a fixed degree are shift invariant, it is sufficient to consider the case $k = 0$ and $i = 0$, that is, the cubic polynomial such that $p_3(j) = f_j$ for $j = -1, 0, 1, 2$. Using the Lagrange interpolation property, we have

$$\begin{aligned}
p_3(1/8) &= L_{-1}(1/8)f_{-1} + L_0(1/8)f_0 + L_1(1/8)f_1 + L_2(1/8)f_2, \\
p_3(3/8) &= L_{-1}(3/8)f_{-1} + L_0(3/8)f_0 + L_1(3/8)f_1 + L_2(3/8)f_2, \\
p_3(5/8) &= L_{-1}(5/8)f_{-1} + L_0(5/8)f_0 + L_1(5/8)f_1 + L_2(5/8)f_2, \\
p_3(7/8) &= L_{-1}(7/8)f_{-1} + L_0(7/8)f_0 + L_1(7/8)f_1 + L_2(7/8)f_2.
\end{aligned}$$

We can find the mask of a quatnary 4-point approximating subdivision scheme:

$$\begin{aligned}
f_{4i}^{k+1} &= -\frac{35}{1024}f_{i-1}^k + \frac{945}{1024}f_i^k + \frac{135}{1024}f_{i+1}^k - \frac{21}{1024}f_{i+2}^k, \\
f_{4i+1}^{k+1} &= -\frac{65}{1024}f_{i-1}^k + \frac{715}{1024}f_i^k + \frac{429}{1024}f_{i+1}^k - \frac{55}{1024}f_{i+2}^k, \\
f_{4i+2}^{k+1} &= -\frac{55}{1024}f_{i-1}^k + \frac{429}{1024}f_i^k + \frac{715}{1024}f_{i+1}^k - \frac{65}{1024}f_{i+2}^k, \\
f_{4i+3}^{k+1} &= -\frac{21}{1024}f_{i-1}^k + \frac{135}{1024}f_i^k + \frac{945}{1024}f_{i+1}^k - \frac{35}{1024}f_{i+2}^k.
\end{aligned}$$

Note: Actually, due to the referee's suggestion, we can derive the masks of this scheme by evaluation at $1/4, 1/2$ and $3/4$ on local cubic interpolation and define the new points at level $k+1$ as a affine combination of 4 points $f_{i-1}^k, f_i^k, f_{i+1}^k, f_{i+2}^k$. That is, we interpolate the data $(j, f_j^k), j = i-1, i, i+1, i+2$ by cubic polynomial p_3 satisfying

$$p_3(j) = f_j^k, \quad j = i-1, i, i+1, i+2$$

and predict

$$f_{4i+1}^{k+1} = p_3\left(i + \frac{1}{4}\right), f_{4i+2}^{k+1} = p_3\left(i + \frac{1}{2}\right), f_{4i+3}^{k+1} = p_3\left(i + \frac{3}{4}\right).$$

We can find the mask of a quaternary 4-point interpolating subdivision scheme:

$$\begin{aligned} f_{4i}^{k+1} &= f_i^k, \\ f_{4i+1}^{k+1} &= -\frac{7}{128}f_{i-1}^k + \frac{105}{128}f_i^k + \frac{35}{128}f_{i+1}^k - \frac{5}{128}f_{i+2}^k, \\ f_{4i+2}^{k+1} &= -\frac{1}{16}f_{i-1}^k + \frac{9}{16}f_i^k + \frac{9}{16}f_{i+1}^k - \frac{1}{16}f_{i+2}^k, \\ f_{4i+3}^{k+1} &= -\frac{5}{128}f_{i-1}^k + \frac{35}{128}f_i^k + \frac{105}{128}f_{i+1}^k - \frac{7}{128}f_{i+2}^k. \end{aligned}$$

3. ANALYSIS OF SCHEME

Theorem 3.1. *([9]) Let S be a subdivision scheme with a mask $\mathbf{a}$ and a dilation matrix M. If S converges uniformly, then for every $\gamma \in E$, where E is the cosets of $\mathbb{Z}^s/M\mathbb{Z}^s$, we have*

$$\sum_{\alpha\in\mathbb{Z}^s} a_{\gamma-M\alpha} = 1. \tag{3.1}$$

The general rule of a quatnary approximating subdivision scheme is given by

$$\begin{aligned} f_{4i}^{k+1} &= \sum_{j\in\mathbb{Z}} a_{4j} f_{i-j}^k, \\ f_{4i+1}^{k+1} &= \sum_{j\in\mathbb{Z}} a_{4j+1} f_{i-j}^k, \\ f_{4i+2}^{k+1} &= \sum_{j\in\mathbb{Z}} a_{4j+2} f_{i-j}^k, \\ f_{4i+3}^{k+1} &= \sum_{j\in\mathbb{Z}} a_{4j+3} f_{i-j}^k. \end{aligned}$$

Let us consider subdivision scheme S with a finite mask $\mathbf{a}$. The symbol of a mask $\mathbf{a}$ is the Laurent polynomial.

$$a(z) = \sum_{i\in\mathbb{Z}} a_i z^i.$$

Theorem 3.1 ($M = 4$) shows that the mask $\{a_i\}_{i\in\mathbb{Z}}$ of a convergent quatnary subdivision scheme S satisfies

$$\sum_{j\in\mathbb{Z}} a_{4j} = \sum_{j\in\mathbb{Z}} a_{4j+1} = \sum_{j\in\mathbb{Z}} a_{4j+2} = \sum_{j\in\mathbb{Z}} a_{4j+3} = 1. \tag{3.2}$$

The symbol of a convergent subdivision scheme satisfies

$$a(e^{i\pi/4}) = a(e^{i\pi/2}) = a(e^{3i\pi/4}) = 0 \text{ and } a(1) = 4,$$

and there exists the Laurent polynomial $a_1(z)$ such that

$$a_1(z) = \frac{4z^3}{(1+z+z^2+z^3)} a(z).$$

The subdivision S_1 with symbol $a_1(z)$ is related to S with symbol $a(z)$ by the following theorem.

Theorem 3.2. *Let S denote a subdivision scheme with symbol $a(z)$ satisfying (3.2). Then there exists a subdivision scheme S_1 with the property*

$$df^k = S_1 df^{k-1},$$

where $f^k = S^k f^0$ and $df^k = \{(df^k)_i = 4^k(f^k_{i+1} - f^k_i) : i \in \mathbb{Z}\}$. And S is a uniformly convergent subdivision scheme if and only if $\frac{1}{4}S_1$ converges uniformly to the zero function for all initial data f^0.

$$\lim_{k\to\infty}\left(\frac{1}{4}S_1\right)^k f^0 = 0.$$

Furthermore, S generates C^m-limit functions provided that the subdivision scheme S_1 generates C^{m-1}-limit functions for some integer $m \geq 1$.

Proof. Using Proposition 3.1 and Theorem 3.2 in [5], we have the theorem □

From the refinement rule of S (4 refinement rules)

$$f^{k+1}_i = \sum_{j\in\mathbb{Z}} a_{i-4j} f^k_j,$$

we have

$$\|f^{k+1}_i\| \leq \left(\sum_{j\in\mathbb{Z}} |a_{i-4j}|\right) \max_j \|f^k_j\|,$$

and we can calculate the norm of S:

$$\|S\|_\infty = \max\left\{\sum_{j\in\mathbb{Z}}|a_{4j}|, \sum_{j\in\mathbb{Z}}|a_{1+4j}|, \sum_{j\in\mathbb{Z}}|a_{2+4j}|, \sum_{j\in\mathbb{Z}}|a_{3+4j}|\right\}.$$

If we define the generating function of control point f^k as

$$F^k(z) = \sum_{i\in\mathbb{Z}} f^k_i z^i,$$

then it is easy to satisfy the following relation

$$F^{k+1}(z) = a(z)F^k(z^4).$$

Further, the Laurent polynomial corresponding to the L-iterated rule S^L is given by

$$a^{[L]}(z) = \prod_{j=0}^{L-1} a(z^{4^j}) = \sum_{i\in\mathbb{Z}} a^{[L]}_i z^i,$$

where the scheme corresponding to a^L is a rule mapping f^k to f^{k+L}

$$f_i^{k+L} = \sum_{j\in\mathbb{Z}} a^{[L]}_{i-4^L j} f_j^k.$$

Note that the norm of S^L is as follows:

$$\|S^L\|_\infty = \max\left\{\sum_{j\in\mathbb{Z}} |a^{[L]}_{i-4^L j}|, i = 0, 1, \dots, 4^L - 1\right\}.$$

For the given mask of approximating scheme:

$$\mathbf{a} = \frac{1}{1024}[-21, -55, -65, -35, 135, 429, 715, 945, 945, 715, 429, 135, -35, -65, -55, -21],$$

we have the mask of S_1:

$$\mathbf{a_1} = \frac{4}{1024}[-21, -34, -10, 30, 149, 260, 276, 260, 149, 30, -10, -34, -21],$$

and we get

$$\left\|\frac{1}{4}S_1\right\|_\infty = \max\left\{\frac{324}{1024}, \frac{296}{1024}, \frac{324}{1024}, \frac{340}{1024}\right\} < 1.$$

We have the mask of S_2:

$$\mathbf{a_2} = \frac{4}{256}[-21, -13, 24, 40, 98, 98, 40, 24, -13, -21],$$

and

$$\left\|\frac{1}{4}S_2\right\|_\infty = \max\left\{\frac{64}{256}, \frac{64}{256}, \frac{132}{256}, \frac{132}{256}\right\} < 1.$$

And the mask of S_3 is

$$\mathbf{a_3} = \frac{4}{64}[-21, 8, 37, 16, 37, 8, -21],$$

and we get

$$\left\|\frac{1}{4}S_3\right\|_\infty = \max\left\{\frac{16}{64}, \frac{58}{64}, \frac{16}{64}, \frac{58}{64}\right\} < 1.$$

Hence the proposed scheme has $C^2(\mathbb{R})$.

4. Approximation Order and Support

While the regularity of the limit function for the subdivision scheme is important, another an important issue of subdivision scheme is how to attain the original function as close as possible if a given initial data f^0 is sampled from an underlying function.

Definition 4.1. *Let us consider the initial grid $X_0 = h\mathbb{Z}$ and initial data $f_i^0 = g(ih)$ sampled a enough smooth function g. Let us denote by f^∞ the limit function obtained through subdivision. The subdivision scheme has approximation order p if*

$$|(g - f^\infty)(x)| \leq Ch^p, \quad x \in \mathbb{R}$$

where C is a real constant and independent of h.

As seen in Theorem 4.1 below, the approximation order of a subdivision scheme can be obtained from its polynomial reproducing property.

Theorem 4.1. *([5]) An convergent subdivision scheme that reproduces polynomial π_n has an approximation order of $n + 1$.*

The polynomial reproducing property (cubic in our work) is the starting point of the construction of the masks as formulated in Sections 2. From Theorem 4.1, the proposed scheme has approximation order 4.
Next, we consider the support of the proposed scheme. This is the support of the basic limit function $\phi = S^\infty\delta$ generated by the given control point $f_i^0 = \delta_{i,0}$

Theorem 4.2. *Let S be the proposed 4-point quatnary approximating subdivision scheme with a mask $\mathbf{a}$ as given in Section 2. Then we have*

$$\textit{supp}(\phi) = \textit{supp}(S^\infty\delta) = \left[-\frac{7}{3}, \frac{7}{3}\right].$$

Proof. Choose $f^0 = \{f_i^0 : f_i^0 = \delta_{i,0},\ i \in \mathbb{Z}\}$, and let $S^\infty\delta = \phi$. From the subdivision rule

$$(S^k\delta)_i = \sum_{j\in\mathbb{Z}} a_{i-4j}(S^{k-1}\delta)_j,$$

we have that $\text{supp}(S\delta) = \text{supp}(\mathbf{a}) = [-7, 7]$ and for each $k = 2, 3, \ldots,$

$$\begin{aligned} \text{supp}(S^k\delta) &= \{i \in \mathbb{Z} : i - 4j \in \text{supp}(\mathbf{a}),\ j \in \text{supp}(S^k\delta)\} \\ &= \{i \in \mathbb{Z} : i \in \text{supp}(\mathbf{a}) + 4\text{supp}(S^k\delta)\}. \end{aligned}$$

Thus, $\text{supp}(S^k\delta) = \frac{4^k-1}{3}\text{supp}(\mathbf{a})$. The values $S^k\delta$ are attached to the parameter values 4^{-k}. Hence, the support of the limit function ϕ is

$$\text{supp}(\phi) = \text{supp}(S^\infty\delta) = \left[-\frac{7}{3}, \frac{7}{3}\right],$$

which completes the proof. □

In Table 1 above, the masks of binary 4-point and 6-point schemes are given by

$$\frac{1}{16}[-1, 0, 9, 16, 9, 0, -1],$$

and

$$\frac{1}{256}[3, 0, -25, 0, 150, 256, 150, 0, -25, 0, 3],$$

Scheme	Approximation order	Support(size)	C^n
binary 4-point	4	6	1
binary 6-point	6	10	2
ternary 3-point	2	4	1
ternary 4-point	3	5	2
quatnary scheme	4	14/3	2

Table 1. Comparison of the proposed scheme to binary 4-point and 6-point, and ternary 3-point and 4-point schemes.

respectively. And the masks of ternary 3-point and 4-point schemes are given by

$$\frac{1}{15}[-1, 0, 4, 12, 15, 12, 4, 0, -1],$$

and

$$\frac{1}{99}[-4, -7, 0, 34, 76, 99, 76, 34, 0, -7, -4],$$

respectively. The support of subdivision scheme influence the locality. We choose a quatnary scheme because one of the best way to get a smaller support is to raise arity. This is one of the advantages of the proposal of the scheme. The support of the proposed subdivision scheme is smaller than the support $[-3, 3]$ of binary 4-point and that of ternary 4 point scheme.

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