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A NOTE ON OPTIMAL RECONSTRUCTION OF MAGNETIC RESONANCE IMAGES FROM NON-UNIFORM SAMPLES IN k-SPACE

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ABSTRACT. A goal of Magnetic Resonance Imaging is reproducing a spatial map of the effective spin density from the measured Fourier coefficients of a specimen. The imaging procedure can be done by inverse Fourier transformation or backward fast Fourier transformation if the data are sampled on a regular grid in frequency space; however, it is still a challenging question how to reconstruct an image from a finite set of Fourier data on irregular points in k-space. In this paper, we describe some mathematical and numerical properties of imaging techniques from non-uniform MR data using the pseudo-inverse or the diagonal-inverse weight matrix. This note is written as an easy guide to readers interested in the non-uniform MRI techniques and it basically follows the ideas given in the paper by Greengard-Lee-Inati [10, 11].

1. INTRODUCTION

In Magnetic Resonance Imaging (MRI), the measured signal $s_m{=}s(t_m)$ induced by the proton density $\rho(r)$ is given by the signal equation

$$s(t_m) = \int_{\text{FOV}} \rho(\mathbf{x}) e^{-2\pi i\,\mathbf{k}(t_m)\cdot\,\mathbf{x}} d\mathbf{x} \tag{1.1}$$

where $\rho(\mathbf{x})$ is a L_2-function compactly supported on the field-of-view (FOV) and $\mathbf{k}(t)$ is a trajectory in the Fourier domain, which can be controlled by the gradient waveforms. The original image can be obtained using an inverse Fourier Transform

$$\rho(\mathbf{x}) = \int_{\mathbf{R}^2} \widehat{\rho}(\mathbf{k})\, e^{2\pi i\,\mathbf{k}\cdot\,\mathbf{x}} d\mathbf{k} \tag{1.2}$$

if one has a complete set of data $\widehat{\rho}(\mathbf{k})$ covering whole k-space. Also a unique reconstructed image in a band-limited function space can be obtained via a backward discrete Fourier transformation if a MR machine is operated by the classical sampling method where $\{s_m\}_{m=1}^{M}$ are given on a regular grid points in the k-space.

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For non-uniform sampling patterns, the FFT cannot be directly applied. And the image restoration process becomes finding an inverse of the ill-posed signal equation with the continuous-to-finite Fourier operator, $\mathbf{s} = \mathcal{F}\rho$. We present two linear reconstruction schemes, $\widetilde{\rho} := \mathcal{F}^* W \mathbf{s}$ in this paper. The first reconstruction method uses a pseudo-inverse for the weight matrix W,

$$\widetilde{\rho}(\mathbf{r}) := \mathcal{F}^* \left(\mathcal{F}\mathcal{F}^*\right)^+ \mathbf{s}. \tag{1.3}$$

where $+$ denotes a pseudo-inverse of $M \times M$ matrix and the second method does a diagonal weight matrix,

$$\widetilde{\rho}(\mathbf{r}) = \textstyle\sum_{m=1}^{M} e^{2\pi i\, \mathbf{k}_m \cdot \mathbf{r}} w_m s_m. \tag{1.4}$$

The purpose of this note is to provide an easy guide to graduate students and researchers who become interested in the non-uniform MRI techniques. A brief introduction to the principles of MRI is given in section 2 and section 3 explains mathematical properties of the continuous-to-discrete Fourier operator. The rest of the paper basically follows the basic ideas of the papers by Greengard-Lee-Inati-*et al.* [10, 11] but adds mathematical details to them. Section 4 provides a mathematical framework regarding error analysis and section 5 deals with some numerical implementation issues.

2. Basics of Magnetic Resonance Imaging

Electro-magnetic signal $M(\vec{r}, t)$ generated by hydrogen atoms in a MRI machine is proportional to the proton density $\rho(r)$,

$$M(\vec{r}, t) = M_0 \rho(\vec{r}) e^{-i\omega t} \tag{2.1}$$

where the Larmor frequency ω is a linear function of magnetic field strength H, $\omega = \gamma H$ with the gyromagnetic ratio $\gamma/2\pi = 42.57$ (MHz/Tesla) for ^{1}H atoms. The existence of the Larmor precession has been predicted by L. Landau and E. Lifshitz in 1935 and a technique known as Nuclear Magnetic Resonance (NMR) spectroscopy has been widely used since 1950s in order to exploit the magnetic properties of certain nuclei by isolating each of frequency components from the total signal captured by an antenna,

$$S(t) = \int_{\text{FOV}} M(\vec{r}, t) d\vec{r}$$

where FOV stands for the field of view.

Imaging technique which localizes proton density $\rho(\vec{r})$ in space, however, has not been known until late 1970s. A first successful imaging method known as the Echo-Planar Imaging (EPI) technique has been proposed by P. Lauterbur and P. Mansfield in 1977. A MRI machine implementing the EPI method first excites protons only on $z = z_0$ plane and then varies magnetic field strength $H(\vec{r}, t)$ in time and space, so that the angular velocity of the signal is no longer a constant,

$$M(\vec{r}, t) = M_0 \rho(\vec{r}|_{z_0}) e^{-i\phi(t,\vec{r})}, \quad \frac{d\phi}{dt}(t, \vec{r}) = \gamma\, H(\vec{r}, t).$$

Suppose that the magnetic field consists of homogeneous background field and time-varying field induced by gradient coils,

$$H(t) = H_0 + \nabla G(t) \cdot \vec{r}. \tag{2.2}$$

Then the phase of the signal is a function of the gradient field $\nabla G(t)$,

$$\phi(t, \vec{r}) = \gamma H_0 t + \gamma \int_0^t G_x(t')x + G_y(t')y \, dt'$$

and the total signal $S(t)$ can be written as follows,

$$\begin{aligned} S(t) &= M_0 \int_{\text{FOV}} \rho(\vec{r}|_{z_0}) e^{-i\phi(t,\vec{r})} d\vec{r} \\ &= M_0 e^{-i\omega_0 t} \int_{\text{FOV}} \rho(\vec{r}|_{z_0}) e^{-i\gamma \int_0^t (G_x x + G_y y) dt'} dxdy, \\ &= M_0' \int_{\text{FOV}} \rho(\vec{r}|_{z_0}) e^{-i\vec{k}(t)\cdot(x,y)} dxdy, \quad M_0' = M_0 e^{-i\omega_0 t} \end{aligned}$$

where

$$\vec{k}(t) = \gamma \int_0^t \big(G_x(t'), G_y(t')\big) \, dt'. \tag{2.3}$$

Therefore, the signal $S(t)$ can be considered as fourier coefficients of $\rho(\vec{r}|_{z_0})$ at $\vec{k}(t)$,

$$S(t) = M_0' \; FT\left[\rho(\vec{r}|_{z_0})\right] \left(\vec{k}(t)\right) \tag{2.4}$$

and a MRI machine is able to accurately measure the fourier transformation of proton density $\rho(\vec{r}|_{z_0})$ at any sampling points along k-space trajectory by controlling the gradient field $\nabla G(t)$ in time. Readers interested in principles and histories of MRI, please refer to a classical text book by Liang and Lauterbur [14].

The k-space trajectory under the classical sampling method, which is commonly used in clinical MRI machines, covers the sampling points on a uniform regular grid. The reconstruction from data on regular grid is rather straight forward even though the data collection step is very slow. It is an active research area introducing a new trajectory on the k-space [1] with faster acquisition time and reconstructing the original proton density.

3. Continuous-to-discrete Fourier transformation

Let $\rho(\mathbf{x})$ be an L_2(FOV) function defined on FOV $:= (-\frac{1}{2}, \frac{1}{2})^2 \subset \mathbf{R}^2$, zero outside of the field of view. The task of image reconstruction is to produce an image of $\rho(\mathbf{x})$ given a finite set of measurements $\{s_m\}_{m=1}^M$ at the data acquisition points $\mathbf{k}_m$ on the $\mathbf{k}$-space trajectory from the MRI signal equation,

$$s_m = (\mathcal{F}\rho)_m := \int_{\text{FOV}} \rho(\mathbf{x}) e^{-2\pi i \, \mathbf{k}_m \cdot \mathbf{x}} d\mathbf{x} \tag{3.1}$$

where $\mathcal{F}$ is a continuous-to-discrete linear operator. The problem of inverting this linear system to find the continuous function $\rho(\mathbf{x})$ from the finite number of samples is inherently ill-posed, and restriction on the domain of the operator or some type of regularization is required.

It is worth to remark that there is a notably important case for this operator inversion problem. Suppose $\{\mathbf{k}_m\}$ are regular grid points dense enough in $\mathbf{k}$-space and $\rho(\mathbf{x})$ is a band-limited L_2(FOV) function, then $\mathcal{F}$ becomes a self-adjoint operator. Thus one can invert the signal equation without any loss using its adjoint

$$(\mathcal{F}^*\{v_m\})(\mathbf{x}) := \sum_{m=1}^{M} v_m e^{2\pi i\,\mathbf{k}_m\cdot\,\mathbf{x}} \tag{3.2}$$

or numerically using a discrete back-ward Fourier transformation. However if $\{\mathbf{k}_m\}$ are irregularly sampled points, then there is no clear way to define a finite dimensional subspace of L_2(FOV) as a pre-image of the operator $\mathcal{F}$. Such a statement can be considered as a generalization of Shannon's sampling theorem [19], which is yet to be studied [2].

This inversion process can be considered as an approximation of inverse Fourier transformation

$$\rho(\mathbf{x}) = \int_{\mathbf{R}^2} \widehat{\rho}(\mathbf{k})\, e^{2\pi i\,\mathbf{k}\cdot\,\mathbf{x}} d\mathbf{k} \tag{3.3}$$

with finite set of Fourier coefficient data $\{s_m = \widehat{\rho}(\mathbf{k}_m)\}_{m=1}^{M}$. Since there is no clear way to define a proper domain of the operator $\mathcal{F}$, it is common practice to restrict the domain of $\mathcal{F}$ to the range of its adjoint operator $\mathcal{F}^*$,

$$\tilde{\rho}(\mathbf{x}) = \mathcal{F}^*\mathbf{v} = \sum_{m=1}^{M} v_m e^{2\pi i\,\mathbf{k}_m\cdot\,\mathbf{x}} \in range(\mathcal{F}^*) \tag{3.4}$$

It is possible to develop a non-linear scheme which maps the input data $\mathbf{s} := \{s_m\}$ to Fourier coefficients of the reconstruction image $\mathbf{v} := \{v_m\}$, however, we want to focus our attention in this paper to linear schemes in the following form,

$$\widetilde{\rho}(\mathbf{x}) = \sum_{m=1}^{M} e^{2\pi i\,\mathbf{k}_m\cdot\,\mathbf{x}} (W\mathbf{s})_m \tag{3.5}$$

where W is a $M \times M$ matrix. In the following sections, we will discuss mathematical properties of the linear reconstruction scheme with a pseudo-inverse and a diagonal-inverse for the weight matrix, W.

4. Reconstruction Error of Pseudo-inverse and Diagonal-inverse

Suppose we have a signal $\mathbf{s} = \mathcal{F}\rho$ and try to apply an image restoration process defined by weight matrix W,

$$\widetilde{\rho} := \mathcal{F}^* W\mathbf{s}. \tag{4.1}$$

A given image function ρ can be uniquely decomposed as

$$\rho = \mathcal{F}^*\mathbf{v} + \rho_0 \text{ where } \mathbf{v} \in Range(F),\ \rho_0 \in Null(F) \tag{4.2}$$

and the reconstruction error in image space is

$$\begin{aligned}\|\widetilde{\rho}-\rho\|^2_{L^2(\text{FOV})} &= \|\mathcal{F}^*W\mathbf{s}-\mathcal{F}^*\mathbf{v}\|^2+\|\rho_0\|^2\\ &= \|\mathcal{F}^*(W\mathcal{F}\mathcal{F}^*-I)\mathbf{v}\|^2+\|\rho_0\|^2. \end{aligned} \tag{4.3}$$

Therefore, it is not possible to find a weight matrix W which matches the reconstructed $\widetilde{\rho}$ to the original image ρ exactly for any ρ with $\rho_0(\neq 0) \in Null(\mathcal{F})$. Instead of measuring the reconstruction error in image space, we measure the reconstruction error in signal space defined as follows

$$\|\widetilde{\rho}-\rho\|^2_{\mathcal{F}} := \|\mathcal{F}\left(\widetilde{\rho}-\rho\right)\|_2^2 = \|\mathcal{F}\mathcal{F}^*W\mathbf{s}-\mathbf{s}\|_2^2 \tag{4.4}$$

and the average reconstruction error for W,

$$E^2(W) := \frac{\int_{||\mathbf{s}||_2=1}||\mathcal{F}\mathcal{F}^*W\mathbf{s}-\mathbf{s}||_2^2\,d\mathbf{s}}{\int_{||\mathbf{s}||_2=1}\,d\mathbf{s}} = \frac{1}{M}||\mathcal{F}\mathcal{F}^*W-I||_F^2 \tag{4.5}$$

where $||\cdot||_F$ denotes the Frobenious norm for $M\times M$ matrix.

A special case is taking W in the following form,

$$W=\mathcal{M}^+,\quad \mathcal{M}_{mn}=(\mathcal{F}\mathcal{F}^*)_{mn}=\int_{\text{FOV}}e^{2\pi i\,(\mathbf{k}_n-\mathbf{k}_m)\cdot\,\mathbf{x}}d\mathbf{x}=\text{sinc}(\mathbf{k}_n-\mathbf{k}_m) \tag{4.6}$$

where $\text{sinc}(\mathbf{k}) = \frac{sin(\pi k_x)}{\pi k_x}\frac{sin(\pi k_y)}{\pi k_y}$ and the superscript of $\mathcal{M}^+$ denotes the pseudo-inverse [8] of the sinc-matrix $\mathcal{M}$. Note that the reconstructed pseudo-inverse solution $\widetilde{\rho}(\mathbf{x}) = \mathcal{F}^*\mathcal{M}^+\mathbf{s}$ has the smallest reconstruction error in image space defined in (4.3),

$$\|\widetilde{\rho}-\rho\|^2_{L^2(\text{FOV})} = \|\rho_0\|^2 \leq \min_{\overline{\rho}}\{\|\overline{\rho}-\rho\| : \mathcal{F}\overline{\rho}=\mathbf{s}\} \tag{4.7}$$

since

$$(\mathcal{M}^+\mathcal{F}\mathcal{F}^*-I)\mathbf{v}=0 \ \text{ for }\ \mathbf{v}\in Range(F) \tag{4.8}$$

and the solution also has the minimum L_2-norm among all functions satisfying the data consistency requirement $\mathcal{F}\widetilde{\rho}=\mathbf{s}$,

$$\|\widetilde{\rho}\|^2_{L^2(\text{FOV})} = \|\mathcal{F}^*\mathbf{v}\|^2 \leq \min_{\overline{\rho}}\{\|\overline{\rho}\| : \mathcal{F}\overline{\rho}=\mathbf{s}\}. \tag{4.9}$$

The reconstruction error in signal space defined in (4.5) is always zero,

$$\|\widetilde{\rho}-\rho\|^2_{\mathcal{F}} = \|\mathcal{M}\mathcal{M}^+\mathbf{s}-\mathbf{s}\|_2^2 = 0 \ \text{ for }\ \mathbf{s}\in Range(\mathcal{F}) \tag{4.10}$$

although $||\mathcal{M}\mathcal{M}^+-I||_F^2>0$ for a rank deficient matrix $\mathcal{M}$. The discrepancy is caused by the fact that we take the average of reconstruction error on the unit sphere $\{\mathbf{s} : \|\mathbf{s}\|=1\}$ which might be bigger than that on the actual signal space $\{\mathbf{s} : \mathbf{s}=\mathcal{F}\rho,\ \rho\in L_2(\text{FOV})\}$.

Another special case of the linear reconstruction method (4.1) is choosing a diagonal weight matrix W which makes the reconstructed image in the follow form,

$$(W)_{mn}=w_m\delta_{mn},\quad \widetilde{\rho}(\mathbf{r})=\sum_{m=1}^{M}s_m e^{2\pi i\,\mathbf{k}_m\cdot\,\mathbf{r}}w_m. \tag{4.11}$$

where the quadrature weight $\{w_m\}$ should be fixed in order to minimize the average reconstruction error in signal space $E^2(W) = \frac{1}{M}||\mathcal{M}W - I||_F^2$ defined in (4.5). We differentiate the error $\frac{M}{2}E^2(W)$ with respect to w_m,

$$\frac{1}{2}\frac{\partial}{\partial w_m}\sum_{i,j}|\mathcal{M}_{ij}w_j - \delta_{ij}|^2 = \sum_{i,j}\left(\mathcal{M}_{ij}^2 w_j \delta_{jm} - \delta_{ij}\mathcal{M}_{ij}\delta_{jm}\right). \tag{4.12}$$

Therefore, $\frac{\partial}{\partial w_m}E^2(W) = 0$ implies, $\sum_i \mathcal{M}_{im}^2 w_m = \mathcal{M}_{mm}$, or equivalently,

$$w_m = \frac{1}{\sum_n \operatorname{sinc}^2(\mathbf{k}_n - \mathbf{k}_m)} \text{ for all } m. \tag{4.13}$$

These are the optimum weights for density compensation in gridding reconstruction and the best weights to use in the quadrature reconstruction [3, 11, 18],

$$\widetilde{\rho}(\mathbf{r}) = \sum_{m=1}^{M} \frac{s_m e^{2\pi i\,\mathbf{k}_m\cdot\,\mathbf{r}}}{\sum_n \operatorname{sinc}^2(\mathbf{k}_n - \mathbf{k}_m)} \tag{4.14}$$

in the Frobenious norm sense. The average reconstruction error with the optimal weights $\{w_m\}_{m=1}^M$ is given as

$$\begin{aligned} ||\mathcal{M}W - I||_F^2 &= \sum_{n,m}\left(\mathcal{M}_{nm}^2 w_m^2 - 2\delta_{nm}\mathcal{M}_{nm}w_m + \delta_{nm}\right) \\ &= \sum_m\left((\textstyle\sum_n \mathcal{M}_{nm})w_m^2 - 2w_m + 1\right) = \sum_m (1 - w_m). \end{aligned} \tag{4.15}$$

5. Numerical Implementation Issues

The computation of the sum (3.4)

$$\tilde{\rho}(\mathbf{x}) = \mathcal{F}^*\mathbf{v} = \textstyle\sum_{m=1}^M v_m e^{2\pi i\,\mathbf{k}_m\cdot\,\mathbf{x}}$$

with $O(M)$ frequency data and $O(M)$ target points seems to require $O(M^2)$ operations, however, can be computed in $O(M\log M)$ operations using the nonuniform fast Fourier transform (NUFFT) which is now a relatively mature technology [4, 7, 16]. The evaluation of the reconstructed image on a regular grid points using the frequency data at irregular sampling points is referred as a Type-1 NUFFT in the paper by Greengard and Lee [9, 15].

The values $\{w_m\}$ in (4.11) can be considered quadrature weights, and the computation of all of the optimal weights in (4.13)

$$\frac{1}{w_m} = \textstyle\sum_n \operatorname{sinc}^2(\mathbf{k}_n - \mathbf{k}_m).$$

appears to require $O(M^2)$ operations, however, the fast sinc^2-transform described in [10] again reduces the computational cost to $O(M\log M)$. Therefore, the image reconstruction based on the optimal quadrature weights

$$\widetilde{\rho}(\mathbf{r}) = \textstyle\sum_{m=1}^M e^{2\pi i\,\mathbf{k}_m\cdot\,\mathbf{r}} w_m s_m$$

can be computed in $O(M \log M)$ cost using a single application of sinc^2-transform followed by one Type-1 NUFFT evaluation.

There are many attempts to solve the signal equation (3.1) directly with some additional constrains or regularization techniques [5, 13, 17]. The minimum-norm least-squares solution to this problem, denoted by $\widetilde{\rho}(x)$, can be found by applying the pseudo-inverse of the finite-discrete operator $\mathcal{F}$ to the signal. Following the discussion of [8, 20], we write the pseudo-inverse solution $\widetilde{\rho}(x)$ defined in (4.6) with the pseudo-inverse of the sinc-matrix,

$$\widetilde{\rho}(\mathbf{r}) = \mathcal{F}^{+}\mathbf{s} = \mathcal{F}^{*}\mathcal{M}^{+}\mathbf{s}, \tag{5.1}$$

can be computed in two steps,

$$\mathcal{M}\mathbf{a} = \mathbf{s} \quad \text{followd by} \quad \widetilde{\rho}(\mathbf{r}) = \mathcal{F}^{*}\mathbf{a} \tag{5.2}$$

where the second step can be done by a single application of the Type-1 NUFFT evaluation. The matrix $\mathcal{M}$ is symmetric semi-positive definite, however, may be ill-conditioned [20]. Therefore, computation of

$$\mathbf{a} = M^{+}\mathbf{s} \tag{5.3}$$

can be done using the singular value decomposition (SVD) with some regularization techniques at a cost of $O(M^3)$ work or solved iteratively using the conjugate gradient method as suggested in [5]. With aid of the fast sinc-transform described in [10], an iterative solution of (5.3) can be obtained in $O(J \cdot M^2)$ computational cost where J denotes the number of iterations and M does the number of data points in k-space.

Image reconstruction process using either pseudo-inverse or diagonal-inverse requires a Type-1 NUFFT application for the final image generation and both benefit from the fast sinc^2-transformation for optimal weights and the fast sinc-transformation for the iterative solution of the signal equation.

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