CONVERGENCE OF NEWTON'S METHOD FOR SOLVING NONLINEAR MATRIX EQUATIONS

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ABSTRACT

Newton's method is efficient numerical methods for solving quadratic matrix equations and matrix polynomials. In this paper we improve the convergence at which the Fréchet derivative is singular for Newton's method and establish an algorithm which guarantee convergence for particular starting matrices.

INTRODUCTION

Nonlinear matrix equations occur in applications and modeling of scientific problems. Here we first consider the quadratic matrix equation which equation can be defined by

$$Q(X) = AX^2 + BX + C = 0, \tag{1}$$

where A, B and C are square matrices. A matrix S satisfying the equation $Q(S) = 0$ is called a solvent.

In this paper we improve the convergence at which the Fréchet derivative is singular for Newton's method and establish an algorithm which guarantee convergence for particular starting matrices. We consider first the generalized Sylvester equation

$$A_1XB_1 + A_2XB_2 = E \tag{2}$$

where A_1, A_2, B_1, B_2 and D are $n \times n$ matrices with real elements. Mao et al [1] presented an efficient iterative method for finding the generalized centro-symmetric solution of matrix equation $CXD = F$, where $C, D, F \in \mathbb{R}^{n\times n}$. We second consider the generalized Sylvester equation

$$\sum_{i=1}^{m} C_iXD_i = G \tag{3}$$

where $C_i, D_i, G \in \mathbb{R}^{n\times n}$, for all i. Peng [2] presented an iterative method for solving equation $\sum_{j=1}^{l} A_jX_jB_j = H, \quad \text{where} \quad A_j, B_j \in \mathbb{R}^{n\times n}$, for all j, over bisymmetric matrix group $[X_1, X_2, \cdots, X_l]$. They have been proved that the iteration method can be terminated within finite iteration steps for any initial matrix. Both iterative methods can be applied to find symmetric and generalized centro-symmetric and bisymmetric solution of equation (2) and (3).

Davis [3], [4] considered Pure-Newton's method and Higham and Kim [5], [6] incorporated the exact line searches into Newton's methods for solving the quadratic matrix in equation (1).

Kratz and Stickel [7] considered Newton's method and Dennis, Jr.,Traub and Weber [8], [9] suggested Berlloulli's iteration for solving matrix polynomials which has the form

$$P_0X^m + P_1X^{m-1} + \cdots + P_m = 0 \tag{4}$$

where $P_0, P_1, \cdots, P_m, X \in \mathbb{R}^{n\times n}$. We show how to integrate the algorithm for the equation (2) into Newton's method for finding symmetric and generalized centro-symmetric and then bisymmetric solutions of the quadratic matrix equation (1). Finally, we extend this algorithm to solve matrix polynomial and illustrate Newton's method with some numerical examples.

NUMERICAL RESULTS

Now we consider the following examples to illustrate our approach. The first example is

$$Q(X) \equiv X^2 + \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix} X + \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix} = 0. \tag{5}$$

Then, $Q(X) = 0$ has infinitely many real solvents, which have the forms

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}, \quad \begin{bmatrix} 3-z & z+1 \\ -z+4 & z \end{bmatrix}, \quad \begin{bmatrix} -5-z & -1-z \\ z+4 & z \end{bmatrix}. \tag{6}$$

where $z \in \mathbb{C}$. Here it is easy to see that the previous three values is symmetric and generalized centro symmetric matrices.

Table 1 explain in this case, both Pure Newton's Method and New Newton,s Method determining to solvent.

No.iteration	Relative residual of Pure Newton's Method	Relative residual of Newton's Method
1	5.30e-001	5.30e-001
2	3.13e-001	3.13e-001
3	8.95e-002	8.95e-002
4	1.48e-002	1.48e-002
5	9.35e-004	9.35e-004
6	5.09e-006	5.09e-006
7	1.55e-010	1.55e-010
8	2.98e-017	5.23e-017

Table 2. Comparison of relative residual for with Pure Newton's Method and Newton's Method.

We consider next quadratic matrix equation. Suppose

$$Q(X) \equiv \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} X^2 + \begin{bmatrix} 0 & -4 \\ 0 & -4 \end{bmatrix} X + \begin{bmatrix} -176 & 144 \\ -176 & 144 \end{bmatrix} = 0. \tag{7}$$

Then, $Q(X) = 0$ has real solvents.

From Table 2 we can see that, if Fréchet derivative is singular, then Pure Newton's Method does not work.

No.iteration	Relative residual of Pure Newton's Method	Relative residual of Newton's Method
1	fail	5.83e-001
2		4.29e-001
3		1.93e-001
4		3.31e-002
5		9.34e-004
6		7.40e-007
7		4.64e-013
8		5.17e-017

Table 2. Fréchet derivative is singular case.

Finally, we give two examples for matrix polynomials.

Let

$$P(X) \equiv X^3 + \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix} X^2 + \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix} X = 0.$$

Then, $P(X) = 0$ has bisymmetric and symmetric and generalized centro-symmetric solvents.

Let

$$P(X) \equiv X^4 + \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix} X^3 + \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix} X^2 + X + \begin{bmatrix} 0 & -4 \\ 0 & -4 \end{bmatrix} = 0.$$

Then, $P(X) = 0$ has bisymmetric and symmetric and generalized centro-symmetric solvents.

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