Block Iterative Solvers for Higher Order Finite Volume Methods

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ABSTRACT

Recently, new higher order finite volume methods(FVM) were introduced by Cai, Douglas and Park[5], where the linear system derived by the hybridization with Lagrange multiplier satisfying the flux consistency condition is reduced to a linear system for pressure variable by an appropriate quadrature rule. We study the convergence of iterative solver for this linear system. The conjugate gradient(CG) method is a natural choice to solve the system, but it seems slow, possibly due to the non-diagonal dominance of the system. In this paper, we propose block iterative methods with a reordering scheme to solve the linear system derived by the higher order FVM and prove their convergence. With a proper ordering, each block subproblem can be solved by fast methods such as multigrid(MG) method. The numerical experiments show that these block iterative methods are much faster than CG.

1 INTRODUCTION

The mixed finite element method(MFEM) has been used and analyzed extensively for many years by now[2–4,8,11,13,17,20]. It provides an efficient way of computing the velocity/flow of a flux variable. The accurate computation of flow is not only important by itself, but it affects the computation of other variables also. In porous media problems, for example, the concentration of some chemical is transported by a moving fluid. Hence it is essential to find accurate velocity of the fluids[10,19]. By introducing the Darcy's velocity $\mathbf{u} = -\kappa\nabla p$ as a new variable and writing a system of partial differential equation involving two variables $\mathbf{u}$ and p simultaneously, we obtain the so-called mixed formulation. It satisfies the local conservation law and so does the discrete counterpart, thus one can expect a better numerical approximation for flow variable. On the other hand, the mixed method has a drawback since it leads to a saddle point problem, hence the linear system which is very expensive to solve. But this shortcoming can be resolved by introducing the Lagrange multipliers to relax the continuity of normal components of the velocity on the edges of the mesh[1,9]. Then the resulting system is symmetric positive definite so that many efficient iterative solvers are applicable.

Another effective way of solving fluid flow equations advocated by engineering community is the finite volume method(FVM), where unknown variables are located only inside of each cell [6,12]. FVM has some advantages of local mass conservation and easy adaptivity to the geometry of the domain. Also the resulting linear system is often (symmetric) positive definite. In general, the FVM formulation is obtained by integrating on certain volume to conserve the physical quantities such as mass, momentum, or energy. Since it satisfies conservation laws on arbitrary volume, it is widely used in many applications, for instance, convection-diffusion equations, Stokes equations, or Euler equations, etc.[6,7,15,16,18]. In FVM, one usually assigns unknowns at the center of each element and use piecewise constant basis functions to obtain the discrete equation. In case of elliptic problems on rectangular grid, we obtain the cell-centered finite difference method[12] which can also be derived from Raviart-Thomas space of the lowest order by applying a certain quadrature rule[20,21]. Most of these are basically low order methods.

Recently, Cai, Douglas and Park [5] proposed a new higher order FVM based on MFEM, which has some advantages of both the MFEM and FVM, in the sense that it is derived from MFEM, while the unknowns are located inside the cell. They start hybridizing MFEM formulation using Lagrange multipliers, then use a suitable quadrature rule to simplify the linear system. An appropriate quadrature rule plays a key role in eliminating the flux variable, so that they obtain an explicit system for scalar and the Lagrange multipliers. Then the Lagrange multipliers are canceled from the resulting system by a judicious choice of basis functions. It allows the original system of three variables(velocity, pressure and Lagrange multiplier) to reduce to a symmetric positive definite algebraic equation for pressure variable only. The velocity is recovered easily from the computed scalar variable.

On the other hand, the solution process of the resulting algebraic system obtained from this higher order FVM is not well-known. The purpose of this paper is to propose and prove the convergence of some iterative methods for solving the system arising from this higher order FVM scheme. Unlike most lower order methods, this system is *not diagonally dominant*. In fact, the ratios between the diagonal elements and the off diagonal elements range from $O(h)$ to $O(1/h)$!(See section 3.) Hence we cannot expect fast convergence of most classical iterative methods. In our scheme the variables are ordered in such a way that we get a nice block matrix which can be solved by block iterative algorithm. Here, each block has a structure similar to the low order method, so that fast algorithms such as conjugate gradient(CG) or multigrid(MG) can be used for the subproblem. We prove the convergence of block iterative methods such as Jacobi, Gauss-Seidel and SOR with properly ordered nodes .

An outline of this paper is as follows. In the next section, we briefly review higher order FVM of [5]. In section 3, we first investigate the structure of the matrix in detail. We then, propose block iterative methods based on certain orderings of variables. We prove the convergence of block Jacobi, Gauss-Seidel, and SOR methods using the theory developed by D. Young[22]. In the final section, we present numerical experiments. Our block iterative methods are indeed much faster than pointwise counterpart or CG.

2 SECOND ORDER FINITE VOLUME METHODS

In this section, we briefly describe higher order FVM introduced by Douglas et al.[5]. Let Ω be a bounded polygonal domain in $\mathbb{R}^n$ with boundary $\partial\Omega$. We consider the homogeneous

Dirichlet problem

$$\begin{cases} -\nabla \cdot (\kappa \nabla p) = f \text{ in } \Omega, \\ \qquad\qquad p = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where κ is a bounded and symmetric positive definite matrix. Introducing the flow variable $\mathbf{u} = -\kappa\nabla p$, we obtain a standard mixed form of (2.1) as follows:

$$\begin{cases} \mathbf{u} + \kappa\nabla p = 0 \;\; \text{in } \Omega, \\ \quad \mathrm{div}\mathbf{u} = f \;\; \text{in } \Omega, \\ \qquad\; p = 0 \;\; \text{on } \partial\Omega. \end{cases}$$

Let

$$\mathbf{V} = \mathbf{H}(\mathrm{div};\Omega) = \left\{ \mathbf{v} \in (L^2(\Omega))^n : \mathrm{div}\mathbf{v} \in L^2(\Omega) \right\},$$

and set $W = L^2(\Omega)$. Then the weak form of problem (2.1) is

$$\begin{cases} (\kappa^{-1}\mathbf{u}, \mathbf{v}) - (p, \mathrm{div}\mathbf{v}) = 0, \qquad \forall \mathbf{v} \in \mathbf{V}, \\ \qquad\qquad (\mathrm{div}\mathbf{u}, q) = (f, q), \; \forall q \in W. \end{cases} \tag{2.2}$$

Assume that we have some triangulation $\mathcal{T}_h$ of Ω into triangles or rectangles. Also, suppose we have some mixed finite element subspaces $\mathbf{V}_h \subset \mathbf{V}$, $W_h \subset W$ associated with this triangulation. Then the finite dimensional problem for (2.2) is: Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ such that

$$\begin{cases} (\kappa^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \mathrm{div}\mathbf{v}_h) = 0, \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ \qquad\qquad (\mathrm{div}\mathbf{u}_h, q_h) = (f, q_h), \; \forall q_h \in W_h. \end{cases} \tag{2.3}$$

This leads to a saddle point problem which is difficult to solve. A common procedure to avoid it is to introduce a Lagrange multiplier to hybridize the system. The localized version of (2.3) using Lagrange multiplier is: Find $(\mathbf{u}_h, p_h, \lambda_h) \in \tilde{\mathbf{V}}_h \times W_h \times \Lambda_h$ such that for any $K \in \mathcal{T}_h$,

$$\begin{aligned} (\kappa^{-1}\mathbf{u}_h, \mathbf{v}_h)_K - (p_h, \mathrm{div}\mathbf{v}_h)_K + < \mathbf{v}_h \cdot \mathbf{n}_K, \lambda_h >_{\partial K\setminus\partial\Omega} = 0, \; &\forall \mathbf{v}_h \in \mathbf{V}_h(K), \\ (\mathrm{div}\mathbf{u}_h, q_h)_K = (f, q_h)_K, \qquad &\forall q_h \in W_h(K), \\ \sum_{K\in\mathcal{T}_h} < \mathbf{u}_h \cdot \mathbf{n}_K, \mu >_{\partial K\setminus\partial\Omega} = 0, \qquad &\forall \mu \in \Lambda_h. \end{aligned} \tag{2.4}$$

Now we describe higher order Finite Volume Methods(FVM) introduced by Douglas et al.[5]. From here on, we assume Ω is the unit square $[0,1]^2$ and it is partitioned into N^2 elements by $N-1$-vertical and horizontal lines. Then the size of the element, h, is $1/N$. A typical element is denoted by $K_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $i, j = 1, \cdots, N$. We shall consider $BDFM_{[k+1]}$ or

$RT_{[k]}$ for $k \geq 1$, and for simplicity we present the case when $k = 1$. For $k > 1$ similar treatment applies.

The matrix form of (2.4) can be written as

$$\begin{cases} \mathbf{AU} + \mathbf{BP} + \mathbf{C}\lambda = \mathbf{O}, \\ \mathbf{B}^t\mathbf{U} = \mathbf{F}, \\ \mathbf{C}^t\mathbf{U} = \mathbf{O}, \end{cases} \tag{2.5}$$

where $\mathbf{A}$ is a block diagonal matrix. For example, if $BDFM_{[2]}$(resp. RT_1) is chosen for $\mathbf{V}_h$ then $\mathbf{A}$ is a block diagonal with the blocks of the 10×10(resp. 12×12) entries, where each block corresponds to an element K. To solve the system (2.5), we eliminate $\mathbf{U}$ first. Then

$$\begin{cases} -\mathbf{B}^t\mathbf{A}^{-1}(\mathbf{BP} + \mathbf{C}\lambda) = \mathbf{F}, \\ -\mathbf{C}^t\mathbf{A}^{-1}(\mathbf{BP} + \mathbf{C}\lambda) = \mathbf{O}. \end{cases}$$

There are two approaches of solving this system. One is to eliminate $\mathbf{P}$ first, so that we obtain a linear algebraic system for λ. This is a well-known method which was studied by Arnold and Brezzi[1]. In this method, one computes λ first, then recover $\mathbf{P}$ and $\mathbf{U}$ by a post-processing technique. On the other hand, Douglas et al. eliminate λ first, so that we obtain a system for $\mathbf{P}$. The advantage of this approach is that one can solve for $\mathbf{P}$ directly and $\mathbf{U}$ can be obtained from a simple explicit formula. There is no need to compute λ.

Thus, we eliminate λ using the Schur complement to obtain

$$\left\{\mathbf{B}^t\mathbf{A}^{-1}\mathbf{C}(\mathbf{C}^t\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}^t\mathbf{A}^{-1}\mathbf{B} - \mathbf{B}^t\mathbf{A}^{-1}\mathbf{B}\right\}\mathbf{P} = \mathbf{F}. \tag{2.6}$$

In general, (2.6) is very complicated and expensive to solve. In order to avoid its complication, they propose new basis functions and some quadrature rule to diagonalize A so that they obtain an explicit equation for $\mathbf{P}$.

3 GROUP ITERATIVE METHODS

The linear equation obtained in the previous section has the form

$$\mathcal{M}\mathbf{p} = \mathbf{f}, \tag{3.1}$$

where ordering of variables is naturally chosen as $\mathbf{p} = (\mathbf{p}_1, \cdots, \mathbf{p}_{N^2})$, where $\mathbf{p}_l = (p_l^0, p_l^x, p_l^y),\ (l = 1, \cdots, N^2)$ represents variables corresponding to the l-th element. Here, $\mathcal{M}$ is $N^2 \times N^2$ block matrix whose blocks are 3×3 matrices.

We see the matrix (3.1) is symmetric positive definite, but *non-diagonally dominant*. The diagonal elements of third rows of the diagonal blocks($M_{i,i}$) are of order h^2 while some off diagonal elements of the same row is of order h. Hence we cannot expect a convergence of classical iterative methods such as Jacobi method, Gauss-Seidel method. CG may work, but it seems hard to devise efficient preconditioners. However, with a proper reordering, we can use block Jacobi, Gauss-Seidel method where fast solvers such as MG can be used for solving the subproblems. Also, CG Preconditioned by MG is straightforward to apply.

The following results in [22, Chap. 5, p. 142] can be applied to show the convergence.

Theorem 3.1 *If A is a π-GCO-matrix such that D_π is nonsingular, then*
(a) If μ_π is any eigenvalue of B_π of multiplicity p, then $-\mu_\pi$ is also an eigenvalue of B_π of multiplicity p.
(b) $S(\mathscr{L}_\pi) = (S(B_\pi))^2$.
(c) λ_π satisfies $(\lambda_\pi + \omega_\pi - 1)^2 = \omega_\pi^2 \mu_\pi^2 \lambda_\pi$ for some eigenvalue μ_π of B_π if and only if λ_π satisfies $\lambda_\pi + \omega_\pi - 1 = \omega_\pi \mu_\pi \lambda_\pi^{1/2}$ for some eigenvalue μ_π of B_π.
(d) If λ_π satisfies either, and hence both of the relations above, then λ_π is an eigenvalue of $\mathscr{L}_{\omega_\pi}$.
(e) If λ_π is an eigenvalue of $\mathscr{L}_{\omega_\pi}$, then there exists an eigenvalue μ_π of B_π such that above relation hold.

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