

# A Positivity Preserving Central-Upwind Scheme for Chemotaxis and Haptotaxis Models

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## ABSTRACT

In this talk, I will present a new finite-volume method for a class of chemotaxis models and for a closely related haptotaxis model. In its simplest form, the chemotaxis model is described by a system of nonlinear PDEs: a convection-diffusion equation for the cell density coupled with a reaction-diffusion equation for the chemoattractant concentration.

The most common formulation of the chemotaxis model is the Keller-Segel system [1], which can be written in the dimensionless form as:

$$\begin{aligned}u_t + \nabla \cdot (\chi u \nabla v) &= \Delta u, \\v_t &= \Delta v - v + u.\end{aligned}\tag{1}$$

Here,  $u(x, y, t)$  denotes the cell density,  $v(x, y, t)$  stands for a chemoattractant concentration, and  $\chi$  is a chemotactic sensitivity constant. It is well-known that solutions of the system (1) may blow up in finite time, see, e.g., [2,3] and the references therein. This blow-up represents a mathematical description of a cell concentration phenomenon that occur in real biological systems, see, e.g., [4–10].

The Keller-Segel model (1) can be generalized to better describe the reality by taking into account several additional factors. For instance, one may consider a more complicated (than just a constant) and a more realistic chemotactic sensitivity function,  $\chi = \chi(u, v)$  as, e.g, in [11–15]. Some other factors, such as growth and death of cells, more accurate terms describing production and uptake of the chemoattractant by cells, presence of food and other chemicals in the system, may also be incorporated into the chemotaxis model, see, e.g., [14,15].

A common property of all existing chemotaxis systems is their ability to model a concentration phenomenon that mathematically results in solutions rapidly growing in small neighborhoods of concentration points/curves. The solutions may blow up or may exhibit a very singular, spiky behavior. In either case, capturing such singular solutions numerically is a challenging problem. Several numerical methods for a simpler version of the Keller-Segel model, [16,17], in which the second equation in (1) has been replaced by the elliptic equation,  $\Delta v - v + u = 0$ , using an assumption that the chemoattractant concentration  $v$  changes over much smaller time scales than the cell density  $u$ . A fractional step numerical method for a fully time-dependent chemotaxis system from [15] has been proposed in [18]. However, the fractional step (operator splitting) approach may not be applicable in the case when the convective part of the chemotaxis system

is not hyperbolic, which is a generic situation as we demonstrate below. Thus, there is a need in accurate and robust numerical methods for general chemotaxis models, especially since many interesting theoretical questions are still completely open due to the lack of reliable numerical tools.

In this talk, I will first provide a simple mathematical explanation on why a naïve numerical approach for the system (1) may not be able to capture its blowing up solutions and then introduce a new method from [19].

The first step in the derivation of the new method is made by adding an equation for the chemoattractant concentration gradient to the original system. I will then show that the convective part of the resulting system is typically of a mixed hyperbolic-elliptic type and therefore straightforward numerical methods for the studied system may be unstable. The new method is based on the application of the second-order central-upwind scheme, [20,21], originally developed for hyperbolic systems of conservation laws, to the extended system of PDEs, and proved to be positivity preserving, which is a very important stability property of the method.

I will illustrate the performance of the new method on a number of two-dimensional problems including the most commonly used Keller-Segel chemotaxis model and its modern extensions as well as to a haptotaxis system modeling tumor invasion into surrounding healthy tissue. The numerical results demonstrate high accuracy, stability, and robustness of the proposed scheme.

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