OPTIMAL L^∞- ERROR ESTIMATE OF A FINITE ELEMENT METHOD FOR HAMILTON-JACOBI-BELLMAN EQUATIONS.

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Abstract. The paper is concerned with the finite element approximation of Hamilton-Jacobi-Bellman Equations.We establish optimal L^∞- error estimate using the concept of subsolution and discrete regularity.

1. Introduction

This paper is concerned with the standard finite element approximation of the Hamilton-Jacobi-Bellman equation (HJB)

$$\max_{1\leq i\leq M}(\mathcal{A}^i u - f^i) = 0 \text{ in } \Omega \tag{1.1}$$

where Ω is a bounded open domain of $\mathbb{R}^N$, with boundary $\partial\Omega$ sufficiently smooth, the f^i are given smooth functions and the $\mathcal{A}^i$ are second order, uniformly elliptic operators.

HJB-equations are encountered in several applications, for example in stochastic control the solution of (1.1) characterizes the infimum of the cost function associated to an optimally controlled stochastic switching process without costs for switching. (cf e.g. [1]).

Qualitative studies have witnissed an intense activity in the eighties (cf., e.g.[2], [3], [4]). However, as far as numerical analysis is concerned, it seems that only few works are known in the literature (cf. [5], [7], [8], [10]), ([11]).

In [8], by means of a subsolution method a finite element approximation study was conducted, for the first time, for problem (1.1) but no error estimate was given. In [10], by means of a finite element Bensoussan-Lions algorithm version, a quasi-optimal error estimate in the L^∞ norm was derived.

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Let $a^i(.,.)$ denote the bilinear forms associated with the differential operators $\mathcal{A}^i$. In [3], it is proved that the solution of problem (1.1) is the limit in $C(\bar{\Omega})$, as the parameter k tends to zero, of the solution of the weakly coupled system of quasi-variational inequalities

$$\left\{\begin{array}{l} a^i(u^i, v-u^i) \geqq (f^i, v-u^i) \ \ \forall v \in H^1(\Omega) \\ u^i \leq k + u^{i+1}\,, \ \ v \leq k + u^{i+1} \\ u^{M+1} = u^1 \end{array}\right. \tag{1.2}$$

and that the solution of this system is in $W^{2,p}(\Omega)$, and is the least upper bound of the set of subsolutions, i.e,

$$\left\{\begin{array}{l} a^i(w^i, v) \ \leq \ (f^i, v) \ \ \forall v \in H^1(\Omega);\ v \geq 0 \\ w^i \leq k + w^{i+1} \end{array}\right. \tag{1.3}$$

This characterization turns out to be a powerful tool for the finite element analysis of the HJB equation ([8], [10]), as it transfers successfully to the discrete case.

Denoting by $\mathbb{V}_h$ the finite element space consisting of piecewise linear functions, $\mathbb{A}^i$ the matrix with generic coefficients $a^i(\varphi_l, \varphi_s)$, $F_l^i = (f^i, \varphi_l)$, and $\{\varphi_l\}$, $l = 1, 2, ...m(h)$ the basis of $\mathbb{V}_h$, the discrete counterpart of (1.1) consists of seeking $u_h \in \mathbb{V}_h$ such that

$$\max_{1 \leq i \leq M} (\mathbb{A}^i u_h - F^i) = 0 \tag{1.4}$$

Analogously to the continuous case, it is proved in [8] that the solution of (1.4) is the limit in $C(\bar{\Omega})$, as the parameter k tends to zero, of the discrete system of QVIs

$$\left\{\begin{array}{l} a^i(u_h^i, v-u_h^i) \geqq (f^i, v-u_h^i)\ \forall v \in \mathbb{V}_h \\ v \leq k + u_h^{i+1},\ \ u_h^i \leq k + u_h^{i+1} \\ u_h^{M+1} = u_h^1 \end{array}\right. \tag{1.5}$$

, the solution of such a system is the least upper bound of the set of discrete subsolutions, i.e,

$$\left\{\begin{array}{l} a^i(w_h^i, \varphi_s) \ \leq \ (f^i, \varphi_s) \ \ \forall \varphi_s;\ \ s = 1, ..., m(h) \\ w_h^i \leq k + w_h^{i+1} \end{array}\right. \tag{1.6}$$

and, satisfies the so-called "discrete regularity"

$$\left| a(u_h^i, \varphi_s) \right| \leq C \left\| \varphi_s \right\|_{L^1(\Omega)} \tag{1.7}$$

where C is a constant independent of both k and h.

This new concept of "discrete regularity", introduced by Cortey Dumont in [9], can be seen as the discrete counterpart of the Lewy-Stampachia estimate $\|\mathcal{A}^i u^i\|_\infty \leq C$, extended to the variational form through the $L^\infty - L^1$ duality.

In the present paper, this plays an important role as it permits to replace the irregular obstacles "$k + u_h^{i+1}$" with $W^{2,p}(\Omega)$ regular ones, and therefore preserves the optimal convergence order.

The approximation method stands on the construction of a continuous subsolution denoted $\beta^{(h)} = (\beta^{1,(h)}, ..., \beta^{M,(h)})$ such that:

$$\beta^{i,(h)} \leq u^i \quad \text{and} \quad \left\|\beta^{i,(h)} - u_h^i\right\|_\infty \leq Ch^2 \mid \log h \mid^2 \quad i = 1, 2, ..., M$$

and the construction a discrete subsolution $\alpha_h = (\alpha_h^1, ..., \alpha_h^M)$ such that:

$$\alpha_h^i \leq u_h^i \quad \text{and} \quad \left\|\alpha_h^i - u^i\right\|_\infty \leq Ch^2 \mid \log h \mid^2, \quad i = 1, 2, ..., M$$

In this situation we, respectively, establish the optimal L^∞ convergence order for both the system (1.2) and the HJB equation (1.1), that is,

$$\max_{1 \leq i \leq M} \| u^i - u_h^i \|_\infty \leq Ch^2 \mid \log h \mid^2$$

and

$$\| u - u_h \|_\infty \leq Ch^2 \mid \log h \mid^2$$

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