Discontinuous Galerkin Immersed Finite Element Methods For Interface Problems

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ABSTRACT

In this article, we explore the application of immersed finite element (IFE) to the discontinuous Galerkin (DG) formulation for interface problems because of two main features: (a) the conservation property; (b) the flexibility for mesh refinement. The combination of the IFE and DG ideas generates attractive methods that can solve interface problems with structured mesh and mesh refinement. First we will recall a bilinear IFE space published before. Then we apply this IFE space to the symmetric and non-symmetric GD formulations with penalty. Implementation issues will be discussed, especially those related to the mesh refinement. Numerical examples will be presented to illustrate features of these DG immersed finite element methods.

INTRODUCTION

In this paper, we consider a penalty discontinuous Galerkin method with immersed finite element for solving the following interface problem:

$$\begin{aligned} -\nabla\cdot\Big(\beta\nabla u\Big) &= f, \quad (x,y)\in\Omega, \\ u|_{\partial\Omega} &= g \end{aligned} \tag{1}$$

together with the jump conditions on the interface Γ:

$$\begin{aligned} [u]\,|_{\Gamma} &= 0, \\ \left[\beta\frac{\partial u}{\partial n}\right]|_{\Gamma} &= 0. \end{aligned} \tag{2}$$

Here, see the sketch in Figure 1, without loss of generality, we assume that $\Omega\subset\mathbf{R}^2$ is a rectangular domain, the interface Γ is a curve separating Ω into two sub-domains Ω^-, Ω^+ such that $\overline{\Omega}=\overline{\Omega^-}\cup\overline{\Omega^+}\cup\Gamma$, and the coefficient $\beta(x,y)$ is a piecewise constant function defined by

$$\beta(x,y)=\begin{cases}\beta^-, & (x,y)\in\Omega^-,\\ \beta^+, & (x,y)\in\Omega^+.\end{cases}$$

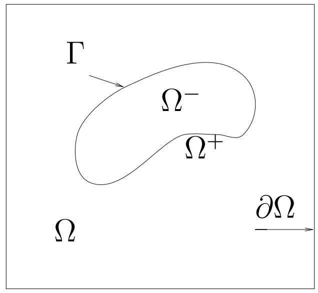

Figure 1. A sketch of the domain for the interface problem.

It is well known that efficiently solving this interface problem is critical in many applications of engineering and sciences, including flow problems,electromagnetic problems, and shape/topology optimization problems, to name just a few.

Interface problem (1) - (2) can be solved by conventional numerical methods, including both finite difference (FD) methods, see [17,28] and references therein, and finite element (FE) methods, see [4,11,13] and references therein, provided that their meshes are tailored to resolve the interfaces, see Figure 2. Otherwise, the lack of smoothness of the exact solution across the interface can make a numerical method not perform as expected or converge at all [9,11,13].

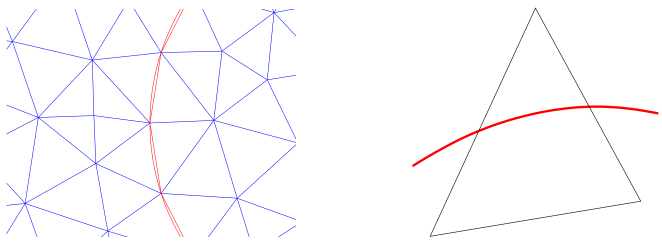

Figure 2. The plot on the left shows how elements are placed along an interface in a standard FE method. An element not allowed in a standard FE method is illustrated by the plot on the right.

Many efforts have been attempted to get rid of this restriction so that interface problems can be solved with meshes independent of the interfaces. In FE formulation, Babuška et al. [5,6,8] developed the generalized and the partition of unity finite element methods. In these methods, the local basis functions in an element are formed by solving the interface problem locally such that these basis functions can capture important features of the exact solution. We note that these basis functions can even be non-polynomials. Exemplary methods in this framework are the partition of unity method and the extended finite element methods (X-FEMs) [10,26,30].

The recently developed immersed finite element (IFE) methods [1,2,12,15,16,19–25,29] fall into the general framework of Babuška and J.E. Osborn [7,8]. These methods solve interface problems by employing local basis functions formed according to the interface jump conditions while their meshes do not have to be conformed with the interfaces. We note that IFE methods do not locally solve the interface problem. The main idea in IFE methods is more similar to that used for the Hsieh-Clough-Tocher macro C^1 element where each local basis function in an element is defined as piecewise polynomials on sub-elements formed the interface such that the required continuity can be satisfied.

All the previously published works are about IFE apply to Galerkin or finite volume formulation. In this article, we explore the application of IFE to discontinuous Galerkin (DG) formulation because of two main features: (a) the conservation property; (b) the flexibility for mesh refinement. First we will recall the bilinear IFE discussed in [16,24]. Then we apply this IFE space to the symmetric [14,18] and non-symmetric [3,27] GD formulations with penalty. Implementation issues will be discussed, especially those related to the mesh refinement. Numerical examples will be presented to illustrate features of these DG immersed finite element methods.

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REFERENCES

1. S. Adjerid and T. Lin. Higher-order immersed discontinuous galerkin methods. *International Journal of Information and Systems Sciences*, 3(4):555–568, 2007.
2. S. Adjerid and T. Lin. p-th degree immersed finite element for boundary value problems with discontinuous coefficients. *Journal of Applied Numerical Mathematics*, (submitted).
3. I. Babuska, C. E. Baumann, and J. T. Oden. A discontinuous hp finite element method for diffusion problems:1d analysis. *Comput. & Math. Appl.*, 37:103–122, 1999.
4. I. Babuška. The finite element method for elliptic equations with discontinuous coefficients. *Computing*, 5:207–213, 1970.
5. I. Babuška, G. Caloz, and J. E. Osborn. Special finite element methods for a class of second order elliptic prolbems with rough coefficients. *SIAM Journal on Numerical Analysis*, 31:945–981, 1994.
6. I. Babuška and J. Melenk. The partition of unity method. *International Journal for Numerical Methods in Engineering*, 40:727–758, 1997.
7. I. Babuška and J. E. Osborn. Generalized finite element methods: their performance and relation to mixed methods. *SIAM J. Numer. Anal.*, 20(3):510–536, 1983.
8. I. Babuška and J. E. Osborn. Finite element methods for the solution of problems with rough input data. In P. Grisvard, W. Wendland, and J.R. Whiteman, editors, *Singular and Constructive Methods for their Treatment, Lecture Notes in Mathematics, #1121*, pages 1–18, New York, 1985. Springer-Verlag.
9. I. Babuška and J. E. Osborn. Can a finite element method perform arbitrarily badly? *Math. Comp.*, 69(230):443–462, 2000.
10. T. Belytschko, N. Moës, S. Usui, and C. Primi. Arbitrary discontinuities in finite elements. *International Journal for Numerical Methods in Engineering*, 50:993–1013, 2001.
11. J. H. Bramble and J. T. King. A finite element method for interface problems in domains with smooth boundary and interfaces. *Adv. Comput. Math.*, 6:109–138, 1996.
12. B. Camp, T. Lin, Y. Lin, and W.-W. Sun. Quadratic immersed finite element spaces and their approximation capabilities. *Advances in Computational Mathematics*, 24:81–112, 2006.
13. Z. Chen and J. Zou. Finite element methods and their convergence for elliptic and parabolic interface problems. *Numer. Math.*, 79:175–202, 1998.
14. M. Delves and C. A. Hall. An implicit matching principle for global element calculations. *J. Inst. Math. Appl*, 23:223–234, 1979.

15. R. E. Ewing, Z. Li, T. Lin, and Y. Lin. The immersed finite volume element method for the elliptic interface problems. *Mathematics and Computers in Simulation*, 50:63–76, 1999.

16. X.-M. He, T. Lin, and Y. Lin. Approximation capability of a bilinear immersed finite element space. *Numer. Methods Partial Differential Equations(to appear)*, 2008.

17. B. Heinrich. *Finite Difference Methods on Irregular Networks*, volume 82 of *International Series of Numerical Mathematics*. Birkhäuser, Boston, 1987.

18. J. Douglas Jr. and T. Dupont. Interior penalty procedures for elliptic and parabolic galerkin methods. *Lecture Notes in Physics*, 58:207–216, 1976.

19. R. Kafafy, T. Lin, Y. Lin, and J. Wang. 3-d immersed finite element methods for electric field simulation in composite materials. *International Journal for Numerical Methods in Engineering*, 64:904–972, 2005.

20. R. Kafafy, J. Wang, and T. Lin. A hybrid-grid immersed-finite-element particle-in-cell simulation model of ion optics plasma dynamics. *Dynamics of Continuous, Discrete and Impulse Systems*, 12:1–16, 2005.

21. Z. Li. The immersed interface method using a finite element formulation. *Applied Numer. Math.*, 27:253–267, 1998.

22. Z. Li, T. Lin, Y. Lin, and R. Rogers. An immersed finite element space and its approximation capability. *Numerical Methods for Partial Differential Equations*, 20(3):338–367, 2004.

23. Z. Li, T. Lin, and X. Wu. New cartesian grid methods for interface problems using finite element formulation. *Numerische Mathematik*, 96(1):61–98, 2003.

24. T. Lin, Y. Lin, R. C. Rogers, and L. M. Ryan. A rectangular immersed finite element method for interface problems. In P. Minev and Y. Lin, editors, *Advances in Computation: Theory and Practice, Vol. 7*, pages 107–114. Nova Science Publishers, Inc., 2001.

25. T. Lin, Y. Lin, and W.-W. Sun. Error estimation of quadratic immersed finite element methods. *Discrete and Continuous Dynamical Systems-Series B*, 7(4):807–823, 2007.

26. N. Moës, J. Dolbow, and T. Belytschko. A finite element method for crack growth without remeshing. *International Journal for Numerical Methods in Engineering*, 46(1):131–150, 1999.

27. J. T. Oden, I. Babuska, and C. E. Baumann. A discontinuous hp finite element method for diffusion problems. *J. Comput. Phys*, 146:491–519, 1998.

28. A. A. Samarskiǐ and V. B. Andreev. *Méthods aux Différences pour Équations Ellipticques*. Mir, Moscow, 1978.

29. S. A. Suater and R. Warnke. Composite finite elements for elliptic boudnary value problems with discontinuous coefficients. *Computing*, 77:29–55, 2006.

30. N. Sukemar, D.L. Chopp, N. Moës, and T. Belytschko. Modling holes and inclusions by level set in the extended finite-element method. *Computer Methods in Applied Mechanics and Engineering*, 190:6183–6200, 2001.