A Nonlinear Mathematical Model of Blood Flow

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ABSTRACT

In the cardiovascular system, blood flow is under constant interaction with the vessel walls. The changes in vessel diameter are regulated by sympathetic interaction and by local conditions within the blood vessel and organ. In this paper, the velocity field and the associated tangential tension corresponding to the flow of an Oldroyd-B fluid are determined with the orientation stress tensor. The coupled constitutive equations and motion equations form a nonlinear system which is solved numerically.

INTODUCATION

Blood flow to an organ is controlled by the constriction and dilation of vessel walls. The first requirement to studying blood flow is to gain a general understanding of the physiology involved. A brief outline of the circulatory system and cardiac output is provided [1–3] to give the reader a better understanding of how blood is circulated and regulated in the human body. Blood vessels provide a tubular network to channel the blood to every possible region of the body, and the heart creates the pressure required to push blood through the vessels. The walls of an artery are distensible tubes of complex elastic behavior. The diameter of the vessel varies with the pulsating pressure [4]. Being elastic, it also propagates pressure and flow waves generated by the heart at a velocity of magnitude mainly determined by the elastic parameters of the wall and the pressure gradient.

In this work we consider the three-dimensional Oldroyd-B model [5] coupled with the momentum equation and the total stress tensor. The numerical analysis of the Oldroyd-B model with velocity components in three dimensional and the momentum equation will be presented. Numerical results and discussion show that the effect of the orientation stress tensor in blood vessel is considerable, although the Brownian force is sufficiently small.

MATHEMATICAL FORMULATION

The most common constitutive equations characterizing this rheological behavior are divided into two general categories namely Newtonian and non-Newtonian models [6,7]. There are the variants of the non-Newtonian model characterizing shear thinning behavior due to rouleaux dispersion at low shear rate. In this paper we consider the non-Newtonian flow which insured that the viscosity of blood $\mu(h, d)$ is a function of vessel diameter d and hematocrit h.

In the Oldroyd-B model [5], the relation between the shear stress tensor B and the orientation stress tensor A is given by:

$$S+\gamma\left[\frac{DS}{Dt}-\nabla V\cdot S-S\cdot(\nabla V)^T\right]=\mu(h,d)\left[B+\gamma\left(\frac{DB}{Dt}-\nabla V\cdot B-B\cdot(\nabla V)^T\right)\right] \\ -gA+C_1\left(gA-\frac{C_2}{\mu(h,d)^2}I\right), \tag{1}$$

where D/Dt is the material derivative, V is the velocity of the fluid, C_1, C_2, g, γ are constants, $B=\nabla V+(\nabla V)^T$, and $S=\mu B+gA$. The mechanical behavior of non-Newtonian fluids can be modeled by several constitutive equations. The continuity and the momentum equations couples with the orientation stress tensor for time-dependent compressible flow is given by:

$$\begin{aligned} &\nabla\cdot V=0,\\ &\rho(\frac{\partial V}{\partial t}+V\cdot\nabla V)=\nabla\cdot\sigma,\\ &\sigma=-pI+\mu(h,d)(\nabla V+(\nabla V)^T)+gA,\\ &\frac{\partial A}{\partial t}+(\vec{v}\cdot\nabla)A-A\cdot\nabla V-(\nabla V)^T\cdot A=-\frac{1}{\tau}(A-\frac{C_2}{\mu(h,d)^2}I), \end{aligned} \tag{2}$$

where ρ is the density, σ the total stress tensor -pI the isotropic pressure stress tensor, g the elastic modulus (constant). The type of flow to be considered here has the velocity field as follows:

$$V=w(r,t)e_z, \tag{3}$$

where e_z is the unit vector along the z direction of the cylinder coordinates system (r,θ,z). This flow field automatically satisfies the constraint of incompressibility. Since the velocity field is independent of θ and z, the stress field will also be independent of θ and z. Substituting (3) in the Oldroyd-B model (1) and in the second equation of the system (2) and recalling the initial conditions

$$S(r,0)=0=A(r,0), \tag{4}$$

we obtain

$$S_{rr}=S_{r\theta}=S_{\theta\theta}=A_{rr}=A_{r\theta}=A_{\theta\theta}=0,$$

and

$$\begin{aligned} &\rho\frac{\partial w}{\partial t}=-\frac{\partial p}{\partial z}+\frac{\partial S_{rz}}{\partial r}+\frac{S_{rz}}{r}+\frac{g}{r}A_{rz}+g\frac{\partial A_{rz}}{\partial r},\\ &S_{rz}+\gamma\frac{\partial S_{rz}}{\partial t}=\mu(h,d)(\frac{\partial w}{\partial t}+\gamma\frac{\partial^2 w}{\partial r\partial t})+(C_1-g)A_{rz},\\ &S_{zz}+\gamma(\frac{\partial S_{zz}}{\partial t}-2\frac{\partial w}{\partial t}S_{rz})=-2C_1(\frac{\partial w}{\partial r})^2+(C_1-g)A_{zz}-\frac{C_2}{\mu(h,d)^2}, \end{aligned} \tag{5}$$

In the numerical study of these phenomena, some simplifications are in order. By eliminating some terms we can obtain a linear partial differential equation. For a detail discussion about this issue and for some interesting examples we refer the reader to [7–9]

NUMERICAL RESULTS

Numerical simulations are certainly considered important tools for prediction of non-Newtonian phenomena, in particular for blood flow models in revelent geometries. Over the past two decades intensive research and significant progress has been made in this area, mainly for steady and unsteady flows of viscoelastic differential and rate-type models. In order to solve the Oldroyd-B model and the momentum equation, we need to specify boundary conditions for either the stress or the velocity with some simplifications. Here, we consider that the unsteady shear flow depends only on z and t:

$$\vec{v}(\vec{x}, t) = (0, 0, \alpha sin(\omega t)z),$$

where α is the amplitude of the pulsatile component and ω is the frequency. A complete schematic scenario of the blood flow in a vessel is represented in the given figures 1 and 2. For the numerical simulation, we consider only velocity in the z-axis with the concept of fluid-mosaic which shows that the magnitude of all of the components of stress are not similar. Hence, examining all of the results from the present figures, we can estimate the effects of the total stress tensor, the orientation stress tensor, pressure, viscosity, and the nonhomogeneity of blood on the flow phenomena quantitatively in order to validate applicability of the present mathematical model. The results have been presented in the form of the total stress tensor increases with velocity profiles.

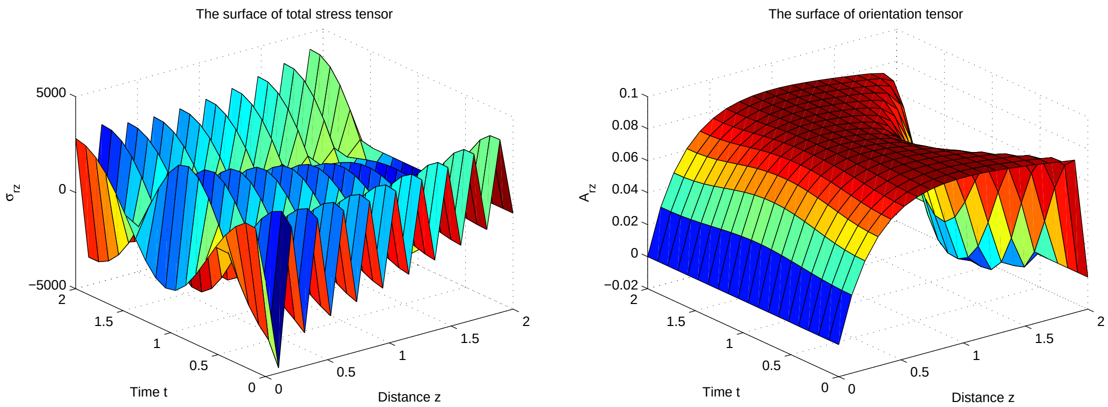

Figure 1. The plots represented the total and orientation stress tensor.

ACKNOWLEDGEMENT

This work was supported by Korea Research Foundation Grant funded by Korea Government (MOEHRD, Basic Research Promotion Fund) (KRF-2005-070-C00010).

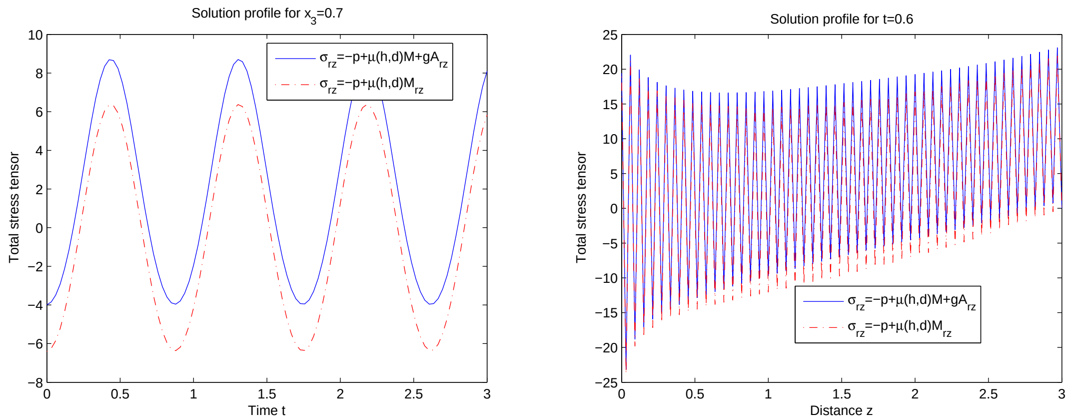

Figure 2. The plots represented the solution profile for x and t.

REFERENCES

1. Fung, Y.C., *"Biomechanics: Circulation"*, Second Edition, Springer-Verlag, New York, 1996.
2. Patrick, H.O. and William, E.C., *"Compressible fluid flow"*, R. R. Donnelley & Sons, 1997.
3. Humphrey, J.D., "Mechanics of the arterial wall: review and directions", *Crit. Review in Bimed. Engineering* Vol. 23, 1995, pp.1-162.
4. Sharan,M. and Aleksander, S.P., "A two-phase model for flow of blood in narrow tubes with increased efficetive viscosity near the wall", *Biorheology* , 38, 2001, pp. 415-428.
5. Oldroyd, J.G., "Non-Newtonian Effects in Steady Motion of Some Idealized Elastico-Viscous Liquids", *Proceedings of R. S. London A.* Vol. 245, 1958, pp. 278-297.
6. Jung, H., Choi, J.W. and Park, C.G., "Asymmetric flows of non-Newtonian fluids in symmetric stenosed artery", *Korea-Australia Rheology Journal* Vol. 16, 2004, pp. 101-108.
7. Rahaman, K.D. and Ramkissoon, H. "Unsteady axial viscoelastic pipe flows", *J. Non-Newtonian Fluid Mechanics* Vol. 57, 1995, pp. 27-38.
8. Zaman, G., Kang, Y.H. and Jung, I.H., "On the flow of an Oldroyd-B fluid with Brownian stress in a blood vessel", Submitted for publication, 2008.
9. Pontrelli, G., "Blood flow through a circular pipe with an impulsive pressure gradient", Math. Mod. & Meth. Appllied Science, 10, 2000, pp. 187-202.