NEW MIXED FINITE ELEMENT FOR ELLIPTIC PROBLEM ON HEXAHEDRAL GRIDS

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ABSTRACT

In this paper, we introduce a new family of mixed finite element on some hexahedral grids. This new element has less degree of freedom than the well-known Raviart-Thomas-Nedelec finite elements, yet enjoys an optimal order approximation for the velocity in L^2-norm. The order of pressure is one less than the velocity. However, we also introduce a local post-processing technique to get an optimal approximation order for the pressure.

MODEL PROBLEM

Let Ω be a bounded polyhedral domain in $\mathbb{R}^3$ with the boundary $\partial\Omega$. We consider the following second order elliptic boundary value problem:

$$\begin{cases} -\mathrm{div}(\kappa\nabla p) = f, \text{ in } \Omega, \\ \qquad\qquad p = 0, \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where f is a given function in $L^2(\Omega)$ and $\kappa = \kappa(\mathbf{x})$ is a symmetric and uniformly positive definite matrix, i.e., there exists two positive constants c_1 and c_2 such that

$$c_1\xi^T\xi \le \xi^T\kappa(\mathbf{x})\xi \le c_2\xi^T\xi, \quad \forall\xi \in \mathbb{R}^3,\ \mathbf{x} \in \overline{\Omega}.$$

MIXED FINITE ELEMENT METHODS

The mixed finite element method has been widely used to obtain accurate approximation of the velocity variables in the porous media problem. In this method, one introduces a new vector variable $\mathbf{u} = -\kappa\nabla p$ and design a finite element method which approximates $\mathbf{u}$ and p simultaneously. Then the problem (1) can be factored to give the first order system

$$\begin{cases} \mathbf{u} + \kappa\nabla p = 0, \text{ in } \Omega, \\ \quad \mathrm{div}\,\mathbf{u} = f, \text{ in } \Omega, \\ \qquad\ p = 0, \text{ on } \partial\Omega. \end{cases} \tag{2}$$

Now we introduce the function spaces

$$\begin{aligned} \mathbf{V} &= H(\mathrm{div}\,,\Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 : \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\}, \\ W &= L^2(\Omega). \end{aligned} \tag{3}$$

The weak form of (2) appropriate for the mixed method is to find $(\mathbf{u}, p) \in \mathbf{V} \times W$ such that

$$\begin{cases} (\kappa^{-1}\mathbf{u}, \mathbf{v}) - (p, \mathrm{div}\,\mathbf{v}) = 0, \qquad \forall \mathbf{v} \in \mathbf{V}, \\ \qquad\qquad (\mathrm{div}\,\mathbf{u}, q) = (f, q),\ \forall q \in W, \end{cases} \tag{4}$$

where $(\cdot, \cdot)$ indicates the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^3$.
Let $\mathscr{T}_h = \{K\}$ be a triangulation of the domain Ω into tetrahedrons, cubes or hexahedrons, and assume that $\mathscr{T}_h = \{K\}$ is regular, i.e. $\mathscr{T}_h = \{K\}$ satisfy the following two conditions:
(i) if h_k is the diameter of K then the quantity $h = max_{K \in \mathscr{T}_h} h_k$ approaches zero.
(ii) if r_k is the radius of the ball inscribed in K, then there exists a constant σ such that $\frac{h_k}{r_k} \leq \sigma$.
Assume that we have some approximating spaces $\mathbf{V}_h \subset \mathbf{V}$ and $W_h \subset W$. Then the mixed finite element approximation $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ is defined as the solution of the equations

$$\begin{cases} (\kappa^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \mathrm{div}\,\mathbf{v}_h) = 0, \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ \qquad\qquad (\mathrm{div}\,\mathbf{u}_h, q_h) = (f, q_h),\ \forall q_h \in W_h. \end{cases} \tag{5}$$

The existence and uniqueness of $(\mathbf{u}_h, p_h)$ follow immediately from the general argument of [4].

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