A NONLINEAR STRUCTURE TENSOR WITH DIFFUSIVITY MATRIX COMPOSED OF IMAGE GRADIENT

Jooyoung HAHN [1] and Chang-Ock LEE [1]

1) *Department of Mathematical Sciences, KAIST, Daejeon, Korea*

Corresponding Author : Jooyoung HAHN, JooyoungHahn@kaist.ac.kr

ABSTRACT

We propose a nonlinear partial differential equation (PDE) for regularizing a tensor which contains the first derivative information of an image such as strength of edges and a direction of the gradient of the image. Unlike a typical diffusivity matrix which consists of derivatives of a tensor data, we propose a diffusivity matrix which consists of the tensor data itself, *i.e.*, derivatives of an image. This allows directional smoothing for the tensor along edges which are not in the tensor but in the image. That is, a tensor in the proposed PDE is diffused fast along edges of an image but slowly across them. Since we have a regularized tensor which properly represents the first derivative information of an image, the tensor is useful to improve the quality of image denoising, image enhancement, corner detection, and ramp preserving denoising. We also prove the uniqueness and existence of solution to the proposed PDE.

INTRODUCTION

The image processing based on partial differential equations (PDEs) has been extensively studied for last 20 years and remarkably successful in problems of computer vision. It starts with an assumption that a gray image is a positive real valued function defined on a rectangular domain even though a digital image is represented by integers from 0 as black to 255 as white. In many low-level topics in computer vision such as image denoising, image enhancement, edge and corner detection, and image segmentation, it is crucial to obtain a regularized derivative information of an image. Since a digital image is represented by integral values, it is difficult to obtain a good approximation of derivatives of the image using a standard finite difference scheme. If a given image has noise, it will be a much harder problem. One of simple solutions is to regularize an image and then differentiate the regularized image. However, in this paper, we use the opposite order of operation; we get derivative information of an image and then regularize it even though the image has noise. In other words, we propose a noble PDE for regularizing a tensor which contains the first derivative information of an image such as strength of edges and a direction of the gradient of the image.

The Perona-Malik (PM) model [1, 2] has been a fundamental frame for adaptive smoothing process based on a nonlinear PDE. Let $I_0\colon \Omega \subset \mathbf{R}^2 \to \mathbf{R}^+$ be an initial noisy image and

h: $\mathbf{R}^+ \to \mathbf{R}^+$ be an weight function with a regularization factor ϵ:

$$h\left(s^2\right) = \frac{1}{\sqrt{\epsilon^2 + s^2}}. \tag{0.1}$$

As time evolves, the PM model generates regularized images $I(x,t)$ which satisfy the PDE

$$\begin{aligned} \frac{\partial I}{\partial t}(x,t) = \nabla \cdot \left(h\left(\left|\nabla I_\sigma\right|^2\right)\nabla I\right) \quad & \text{in} \quad \Omega \times (0, T_1], \\ \nabla I(x,t) \cdot \mathbf{n} = 0 \quad & \text{on} \quad \partial\Omega \times (0, T_1], \\ I(x,0) = I_0(x) \quad & \text{on} \quad \Omega, \end{aligned} \tag{0.2}$$

where $\mathbf{n}$ is a normal vector to $\partial\Omega$ and $I_\sigma \equiv G_\sigma * I$ is the convolution of I with the two-dimensional Gaussian kernel with a standard deviation σ. The PM model uses regularized strength of edges as $|\nabla I_\sigma|$ which makes an adaptive smoothing process. Note that I_σ is an isotropically smoothed image since the Gaussian convolution is equivalent to solve an isotropic linear heat equation. If an initial image is highly noisy, we need to use large σ in order to obtain reliable information of strength of edges. However, the regularized strength of edges with large σ is smeared out and it fails to make effective adaptive smoothing in order to keep edges and corners in an original image. Even though we take small σ, large end time T_1 is needed to obtain a regularized image which has visually small amount of noise. It is not guaranteed that edges and corners are preserved with large end time T_1. Moreover, it is hard to take a proper σ in order to obtain good regularized strength of edges because it is hard to measure an amount of noise in practice.

Weickert [3] proposed the coherence-enhancing diffusion based on a diffusivity matrix which explicitly represents directional smoothing. The diffusivity matrix is obtained by a structure tensor [4–6]

$$G_\sigma * \left(\nabla I_\rho \nabla I_\rho{}^{\mathrm{T}}\right), \tag{0.3}$$

where T is the transpose and $(G_\sigma * M)_{ij} = G_\sigma * m_{ij}$ for a matrix $M = (m_{ij})$. The noise scale ρ is determined by an amount of noise in an initial image and the integration scale σ reflects a size of neighborhood for a local structure analysis. The structure tensor has a remarkable feature in obtaining regularized strength of edges when ρ is taken small enough to compute ∇I_ρ. While the regularized strength of edges in the PM model is obtained by regularizing an image and then differentiating the regularized image, it is obtained in a structure tensor by differentiating an image and then regularizing derivatives of the image. Changing the order of two operations, the structure tensor has better representation of flow-like structures in an image [3]. However, it still has the same problem of choosing a value σ as in (0.2) when an initial image is highly noisy.

From two PDE-based image denoising models in the above, we notice that, for the purpose of adaptive smoothing of an image, it is crucial to obtain regularized strength of edges which properly represents local structures of the image such as edges and corners robust to a change of an amount of noise in the image. Moreover, if we have a regularized direction of the gradient of an image which properly represents the orthogonal orientation to edges in the image, it will improve the quality of image denoising based on an anisotropic diffusion.

In this paper, we propose a nonlinear PDE for regularizing a tensor which contains the first derivative information of an image such as strength of edges and a direction of the gradient of the image. Unlike a typical diffusivity matrix which consists of derivatives of a tensor data [7–13], we propose a diffusivity matrix which consists of the tensor data itself, *i.e.*, derivatives of an image. This allows directional smoothing of a tensor along edges which are not in the tensor but in the image. That is, the tensor is diffused fast along edges of an image but slowly across them. It explains that the proposed PDE generates regularized tensors which adapt to the first derivative information of an image as time evolves. Moreover, the regularized tensor is used to improve the quality of image denoising, image enhancement, corner detection, and ramp preserving denoising.

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