FINITE ELEMENT PRECONDITIONING ON SPECTRAL ELEMENT METHODS FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

The spectral element discretization to solve a second-order uniformly elliptic partial differential equation leads to a linear equation which needs efficient iterative methods such as Schwarz-based methods, preconditioning techniques related to multigrid methods. This is because such linear systems have large condition numbers usually dependent on the mesh sizes h and degrees N of polynomials adopted by spectral element methods. Hence it is very important to provide an optimal preconditioner in the sense that the preconditioned system has small condition numbers which are not dependent on h and N.

We consider an elliptic problem

$$Lu := -\nabla \cdot p\nabla u + qu = f \quad \text{in} \quad \Omega \tag{1}$$

with boundary conditions

$$u = 0 \quad \text{on} \quad \Gamma_D \quad \text{and} \quad \mathbf{n} \cdot \nabla u = 0 \quad \text{on} \quad \Gamma_N \tag{2}$$

where $\partial\Omega = \Gamma_D \cup \Gamma_N$ and Γ_D is a nonempty set. Further, we assume that $0 < p \leq \infty$ and $q \geq 0$ are nonnegative smooth bounded function on Ω.

For the problem (1)-(2), we implement the spectral element discretizations using Chebyshev-Gauss-Lobatto/Legendre-Gauss-Lobatto nodes and present preconditioners using finite element methods at same nodes for each discretization, and also show its optimality or scalability, respectively.

Let ξ^g be the Gauss-Lobatto nodes: denoting $g = c$ for the CGL case and $g = \ell$ for LGL case. Let $\mathcal{V}^g_{h,N}, \mathcal{P}^g_{h,N}$ be the space of Lagrange piecewise linear functions and of the piecewise polynomials based on ξ^g, respectively.

Then the main tool is following as:

Theorem 1 *There are positive constants c and C independent of N and h such that for all $\phi \in \mathcal{V}^g_{h,N}$*

$$c|\mathcal{I}^g_{h,N}\phi|_1 \leq |\phi|_1 \leq C|\mathcal{I}^g_{h,N}\phi|_1, \tag{3}$$

where $\mathcal{I}^g_{h,N}$ is the interpolation operator based on ξ^g, $g = c$ or ℓ:

$$(\mathcal{I}^g_{h,N}v)(\xi^g_\mu) = v(\xi^g_\mu) \quad \textit{for} \quad v \in C(I).$$

Theorem 2 *There are positive constants c and C independent of N and h such that*

$$c\|\mathcal{I}^{\ell}_{h,N}\phi\|_0 \le \|\phi\|_0 \le C\|\mathcal{I}^{\ell}_{h,N}\phi\|_0 \quad \textit{for} \quad \phi \in \mathcal{V}^{\ell}_{h,N}. \tag{4}$$

where $\|\cdot\|_0$ is the L^2-norm.

From Theorem 1 and 2, we have

Corollary 3 *for $\phi \in \mathcal{V}^{\ell}_{h,N}$,*

$$\|\mathcal{I}^{\ell}_{h,N}\phi\|_1 \sim \|\phi\|_1. \tag{5}$$

Theorem 4 *There are positive constants c and C independent of N and h such that*

$$c\|\mathcal{I}^{c}_{h,N}\phi\|_1 \le \|\phi\|_1 \le C\|\mathcal{I}^{c}_{h,N}\phi\|_1 \quad \textit{for} \quad \phi \in \mathcal{V}^{c}_{h,N} \tag{6}$$

Theorem 5 *It follows that for all $\psi \in \mathcal{V}^{c}_{h,N}$,*

$$c_N\|\mathcal{I}^{c}_{h,N}\psi\|_0 \le \|\psi\|_0 \le C\|\mathcal{I}^{c}_{h,N}\psi\|_0$$

where C is an absolute positive constant and c_N is dependent only on N not on h.

Based on these facts, we show the optimality of the finite element preconditioner on the LGL nodes to Legendre spectral element discretization for one- and two-dimensional model problem. Also for spectral element method based on CGL nodes to the problem with the constant coefficients, we provide an optimal finite element preconditioner in one-dimensional case and show in analytic and numerical view of points. In this case, two-dimensional preconditioner is scalable owing to Theorem 5.

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REFERENCES

1. M. O. Deville, P. F. Fischer E. H. Mund, *High-order Methods for Incompressible Fluid Flow*, Cambridge Monographs on Applied and Computational Mathematics, Cambridge Univ. Press, Cambridge, UK, 2002.
2. D. Gottlieb, M.Y. Hussaini and S.A. Orszag, *"Theory and Applications of Spectral Methods," in Spectral Methods for Partial Differential Equations*, ed. by R.G. Voigt, D. Gottlieb, M.Y. Hussaini(SIAM-CBMS, Philadelphia), pp 1-54.
3. J. Heys, T. Manteuffel, S. McCormick and L. Olson, "Algebraic Multigrid(AMG) for High-Order Finite Elements", *J. Comput. Phys.*, 204(2005), pp 520-532.
4. S. Parter and E. Rothman, "Preconditioning Legendre spectral collocation approximations to elliptic problems", *SIAM J. Numer. Anal.*, 32(1995), pp 333-385.
5. A. Toselli and O. Widlund, *Domain Decomposition Methods-Algorithms and Theory"*, Springer Series in Computational Mathematics, Springer-Verlag, Berlin Heidelberg (2005).