EXACT CONTROLLABILITY PROBLEMS ON HYPERBOLIC PDEs

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ABSTRACT

The controllability problems for linear hyperbolic equations are studied through terminal-state tracking optimal control problems. Analytic solution formula for the optimal control problem is derived in the form of eigenseries. It is showed that the optimal solution is a solution of approximate controllability problem. Explicit solution formula for the exact controllability problem is also expressed by the eigenseries formula when the target state and the controlled state have matching boundary conditions. We demonstrate by numerical simulations that the optimal solutions approach the target functions.

1 INTRODUCTION

We study both a terminal-state tracking optimal control problem and a exact controllability problem for linear hyperbolic partial differential equations (PDEs) defined over the finite time interval $[0,T] \subset [0,\infty)$ and on a bounded, C^2 (or convex) spatial domain $\Omega \in \mathbb{R}^d$ ($d = 1$ or 2 or 3). Let target functions $W \in L^2(\Omega)$ and $Z \in L^2(\Omega)$ and initial conditions $w \in H_0^1(\Omega)$ and $z \in L^2(\Omega)$ be given. Let $f \in L^2(0,T;L^2(\Omega))$ denote the distributed control. We consider the following optimal control problem: minimize the terminal-state tracking functional

$$\begin{aligned}\mathcal{J}(u,f) =& \frac{T}{2}\int_\Omega |u(T,\mathbf{x}) - W(\mathbf{x})|^2\,d\mathbf{x} + \frac{T}{2}\int_\Omega |u_t(T,\mathbf{x}) - Z(\mathbf{x})|^2\,d\mathbf{x} \\ &+ \frac{\gamma}{2}\int_0^T\int_\Omega |f(t,\mathbf{x})|^2\,d\mathbf{x}dt\end{aligned} \tag{1}$$

(where γ is a positive constant) subject to a hyperbolic PDE

$$\begin{cases} u_{tt} - div[A(\mathbf{x})\nabla u] = f \quad \text{in } Q \equiv (0,T)\times\Omega\,, \\ u|_{\partial\Omega} = 0 \quad \text{in } (0,T)\,, \\ u|_{t=0} = w \quad \text{and} \quad u_t|_{t=0} = z \quad \text{in } \Omega\,. \end{cases} \tag{2}$$

In (2), $A(\mathbf{x})$ is a symmetric matrix-valued, $C^1(\overline{\Omega})$ function that is uniformly positive definite.

The studies have been conducted to explore the terminal-state tracking optimal control problems[10,11,13,14] and inparticular, eigen series solutions to optimal control problem of linear parabolic equation were considered in [17,21]. In the paper, one of our main achievements in optimal control problem is the derivation and justification of analytic solution formula in the form of eigenseries even though the admissible state u and the desired state W have non-matching boundary conditions.

FORMULATION OF OPTIMAL CONTROL AND CONTROLLABILITY PROBLEMS

Functional (1) can be written as

$$\mathcal{J}(u,f) = \frac{T}{2}(\|u(T,\mathbf{x}) - W(\mathbf{x})\|_0^2 + \|u_t(T,\mathbf{x}) - Z(\mathbf{x})\|_0^2) + \frac{\gamma}{2}\int_0^T \|f(t,\mathbf{x})\|_0^2 dt\,. \quad (3)$$

The optimal control problems we study can be concisely stated as

(OP) seek a pair $(\widehat{u},\widehat{f}) \in \mathcal{V}_{ad}$ such that $\mathcal{J}(\widehat{u},\widehat{f}) = \inf_{(u,f)\in\mathcal{V}_{ad}} \mathcal{J}(u,f)$

where the functional $\mathcal{J}$ is defined by (1).

The approximate and exact controllability problems are formulated as follows: The approximate and exact controllability problems are formulated as follows:

(AP-CON) seek a one-parameter set $\{(u_\epsilon, f_\epsilon) : \epsilon > 0\} \subset \mathcal{V}_{ad}$ such that

$$\lim_{\epsilon\to 0} \|u_\epsilon(T) - W\|_0 = 0 \text{ and } \lim_{\epsilon\to 0} \|u'_\epsilon(T) - Z\|_0 = 0\,,$$

and

(EX-CON) seek a pair $(u,f) \in \mathcal{V}_{ad}$ such that $u(T) = W$ and $u'(T) = Z$ in Ω.

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