A Posteriori Error Estimator for Quadrilateral Nonconforming Finite Element Method of Linear Elasticity

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ABSTRACT

In this talk we discuss some a posteriori error estimators for the rotated-$Q1$ nonconforming finite element method of the planar linear elasticity. Hanging nodes are allowed for local mesh refinement. The first error estimator is derived by applying the equilibrated residual method with the equilibrated Neumann data simply given by the local weak residuals of the numerical solution. From this implicit estimator we then derive an explicit estimator which is similar to the one proposed by Dörfler and Ainsworth [1] for the Stokes problem. It is established that all error estimators thus obtained yield guaranteed upper bounds for the true error in the energy-like error measure (up to higher order terms due to data oscillation), and are robust with respect to the Lamé constants.

INTRODUCTION

In this talk we are concerned with numerical approximation of the linear, isotropic elasticity problem described by

$$\begin{cases} -\mu\Delta\boldsymbol{u} - (\mu+\lambda)\nabla(\operatorname{div}\boldsymbol{u}) = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{u}_D & \text{on } \Gamma, \end{cases}$$

where $\boldsymbol{u}$ is the displacement and $\boldsymbol{f}$ is the body force. The standard weak formulation for this problem is to find $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega;\boldsymbol{u}_D) := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v}|_\Gamma = \boldsymbol{u}_D\}$ such that

$$A(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v})_\Omega \qquad \forall \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega),$$

where

$$A(\boldsymbol{u},\boldsymbol{v}) = \mu(\nabla\boldsymbol{u},\nabla\boldsymbol{v})_\Omega + (\lambda+\mu)(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{v})_\Omega.$$

The constants μ and λ are called Lamé constants, and it is assumed that $\mu_1 \le \mu \le \mu_2$ for some $\mu_1, \mu_2 > 0$ and $0 < \lambda < \infty$.

It is well known that standard low-order conforming finite elements suffer from slow convergence when the material becomes nearly incompressible (cf. [2]). In order to avoid this so-called locking phenomenon, we choose the nonconforming finite element approximation of the displacement on quadrilateral grids combined with the reduced integration technique which was proposed in [3]: find $\boldsymbol{u}_h \in (\mathcal{V}_h)^2$ such that $\int_E(\boldsymbol{u}_h - \boldsymbol{u}_D)\,ds = 0$ for boundary edge E, and

$$A_h(\boldsymbol{u}_h,\boldsymbol{v}_h) = (\boldsymbol{f},\boldsymbol{v}_h)_\Omega \qquad \forall \boldsymbol{v}_h \in (\mathcal{V}_{h,0})^2. \tag{1}$$

where P_h denotes the L^2 projection onto the space of piecewise constants on $\mathcal{T}_h$, and

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \mu(\nabla_h \boldsymbol{u}_h, \nabla_h \boldsymbol{v}_h)_\Omega + (\lambda + \mu)(P_h \operatorname{div}_h \boldsymbol{u}_h, P_h \operatorname{div}_h \boldsymbol{v}_h)_\Omega.$$

Since nonconforming finite elements involve much more degrees of freedom than conforming ones of the same order, for the sake of computational efficiency, it is vital to perform adaptive mesh refinement based on a posteriori error estimators which are robust with respect to the Lamé constants; see, e.g., [4–7]. Our approach in a posteriori error analysis of nonconforming finite element methods is to decompose the approximation error into two components, namely, *conforming* and *nonconforming* ones (cf. [8–10]). More precisely, let $\boldsymbol{\xi} \in \boldsymbol{H}^1(\Omega; \boldsymbol{u}_D)$ be the projection of $\boldsymbol{u}_h$ onto the solution space $\boldsymbol{H}^1(\Omega; \boldsymbol{u}_D)$ with respect to the energy inner product, that is, the solution of

$$A(\boldsymbol{\xi}, \boldsymbol{v}) = A_h(\boldsymbol{u}_h, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \tag{2}$$

subject to the Dirichlet boundary condition $\boldsymbol{\xi}|_\Gamma = \boldsymbol{u}_D$. We then decompose the total error as

$$\boldsymbol{u} - \boldsymbol{u}_h = (\boldsymbol{u} - \boldsymbol{\xi}) + (\boldsymbol{\xi} - \boldsymbol{u}_h),$$

and estimate each component in an independent way. This is mathematically equivalent to the previous approach relying on the Helmholtz decomposition of the *gradient of the error* but seems more natural in estimating the nonconforming error $\boldsymbol{\xi} - \boldsymbol{u}_h$.

The conforming error $\boldsymbol{u} - \boldsymbol{\xi}$ is estimated by applying the equilibrated residual method with the equilibrated Neumann data simply given by the local weak residuals of the numerical solution (for each edge E of $T \in \mathcal{T}_h$)

$$\boldsymbol{g}_T|_E = \frac{1}{|E|}\left\{ \int_T (\mu \nabla \boldsymbol{u}_h + (\lambda + \mu) P_h \operatorname{div} \boldsymbol{u}_h \boldsymbol{I}) \nabla \phi_E^{(T)} \, d\boldsymbol{x} - \int_T \boldsymbol{f} \phi_E^{(T)} \, d\boldsymbol{x} \right\},$$

where $\phi_E^{(T)}$ is the local basis function for the nonconforming finite element space. An explicit error estimator similar to the one proposed by Dörfler and Ainsworth [1] for the Stokes problem can be also derived from this implicit estimator. The nonconforming component $\boldsymbol{\xi} - \boldsymbol{u}_h$ is simply estimated by the averaging technique, but the continuous inf-sup condition is crucially used to derive a sharper upper bound for large values of λ. All error estimators thus obtained yield guaranteed upper bounds for the true error in the energy-like error measure (defined below) up to higher order terms due to data oscillation, and are robust with respect to the Lamé constants.

For measuring the approximation error, we adopt (cf. [10])

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_h^2 = \mu \|\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega}^2 + (\lambda + \mu)\|\operatorname{div} \boldsymbol{u} - P_h \operatorname{div}_h \boldsymbol{u}_h\|_{0,\Omega}^2$$

which is not exactly a norm but intermediate between the continuous energy norm $|||\cdot||| := A(\cdot,\cdot)^{1/2}$ and the discrete energy norm $A_h(\cdot,\cdot)^{1/2}$ (which was used in [3] for a priori error analysis). We note, however, that with the introduction of the continuous and discrete pressure variables

$$p = -(\lambda + \mu) \operatorname{div} \boldsymbol{u}, \qquad p_h = -(\lambda + \mu) P_h \operatorname{div}_h \boldsymbol{u}_h,$$

$|||\cdot|||_h$ can be written as

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_h^2 = \mu \|\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega}^2 + \frac{1}{\lambda + \mu}\|p - p_h\|_{0,\Omega}^2.$$

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