Multiscale Mortar Mixed Finite Element Methods For Subsurface Fluid Flows

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ABSTRACT

In this paper, we consider second order linear elliptic and parabolic equations that model single phase Darcy flows in porous media.

First, we study multiscale mortar mixed finite element discretizations for model elliptic problems introduced by Arbogast *et al.*. This approach is based on domain decomposition theory and mortar finite elements. In this method, flux continuity is imposed via a mortar finite element space on a coarse grid scale, while the equations in the coarse elements (or subdomains) are discretized on a fine grid scale.

We extend the method to treat slightly compressible Darcy flows in porous media. Parallel numerical simulations on some multiscale benchmark problems are given to show the efficiency and effectiveness of the method.

We consider an interface problem involving only the mortar pressure which arises from the mortar mixed formulation. The interface formulation is useful in deriving a bound on the error in the mortar space. Moreover, it is the basis for implementation of parallel domain decomposition methods. However, in the previous work, the mortar pressure error was measured in a semi-norm arising from the interface formulation. It applies to the case where the model problem is symmetric positive definite. It should be noted that in many interesting applications, it is important to be able to treat more general problems including nonsymmetric tensor coefficients and/or convection-diffusion equations. Moreover, in the nonlinear case, the use of the superposition principle is not applicable for the error analysis. Therefore, we introduce a new approach for treating interface problem and prove various stability estimates based on inf-sup conditions related to the mortar pressure variable. Optimal fine scale convergence is obtained by an appropriate choice of mortar grid and polynomial space of approximation.

Finally, we discuss recent results treating convection-diffusion equations and slightly compressible Darcy flows using this new approach. This is a joint work with Mary F. Wheeler.

INTRODUCTION

We consider a second order linear elliptic equation that models single phase Darcy flow in porous media. We solve for pressure p and velocity $\mathbf{u}$ satisfying

$$
\begin{aligned}
\mathbf{u} &= -K\nabla p \quad && \text{in } \Omega, && (1)\\
\nabla \cdot \mathbf{u} &= f && \text{in } \Omega, && (2)\\
p &= 0 && \text{on } \partial\Omega, && (3)
\end{aligned}
$$

where $\Omega \subset \mathbf{R}^d$, $d = 2$ or 3, is the domain and K is a symmetric, uniformly positive definite tensor with $L^\infty(\Omega)$ components representing the permeability divided by the viscosity. For simplicity we assume homogeneous Dirichlet boundary conditions.

In this paper, we study multiscale mortar mixed finite element discretizations for model elliptic problems introduced by Arbogast *et al.* [1]. This approach is based on domain decomposition theory and mortar finite elements [2]. In this method, flux continuity is imposed via a mortar finite element space on a coarse grid scale, while the equations in the coarse elements (or subdomains) are discretized on a fine grid scale. In [4], the method is extended to treat slightly compressible Darcy flows in porous media and parallel numerical simulations on some multiscale benchmark problems [5] are given to show the efficiency and effectiveness of the method.

We consider an interface problem involving only the mortar pressure which arises from the mortar mixed formulation. The interface formulation is useful in deriving a bound on the error in the mortar space. Moreover, it is the basis for implementation of parallel domain decomposition methods. However, in the previous work [1], the mortar pressure error was measured in a semi-norm arising from the interface formulation. It applies to the case where the model problem is symmetric positive definite. It should be noted that in many interesting applications, it is important to be able to treat more general problems including nonsymmetric tensor coefficients and/or convection-diffusion equations. Moreover, in the nonlinear case, the use of the superposition principle is not applicable for the error analysis. Therefore, we introduce a new approach for treating interface problem and prove various stability estimates based on inf-sup conditions related to the mortar pressure variable in [7]. Optimal fine scale convergence is obtained by an appropriate choice of mortar grid and polynomial space of approximation.

FORMULATION OF THE METHOD

Let Ω be decomposed into nonoverlapping subdomain blocks Ω_i, so that $\bar{\Omega} = \cup_{i=1}^{n} \bar{\Omega}_i$ and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. Let $\Gamma_{i,j} = \partial\Omega_i \cap \partial\Omega_j$, $\Gamma = \cup_{1 \le i < j \le n} \Gamma_{i,j}$, and $\Gamma_i = \partial\Omega_i \cap \Gamma = \partial\Omega_i \backslash \partial\Omega$ denote interior block interfaces. Let

$$
\begin{aligned}
\mathbf{V}_i &= H(\text{div};\Omega_i), && \mathbf{V} = \bigoplus_{i=1}^{n} \mathbf{V}_i,\\
W_i &= L^2(\Omega_i), && W = \bigoplus_{i=1}^{n} W_i = L^2(\Omega),\\
M_{i,j} &= H^{1/2}(\Gamma_{i,j}), && M = \bigoplus_{1 \le i < j \le n} M_{i,j}.
\end{aligned}
$$

We consider a nonlinear second order parabolic equation that models Darcy flow in porous media:

$$\frac{\partial}{\partial t}\phi\rho_w(p) - \nabla\cdot K\rho_w(p)(\nabla p - g\rho_w(p)\nabla D) = f \qquad \text{in } \Omega\times J, \tag{4}$$

$$p = p(\rho_b) \quad \text{on } \partial\Omega\times J, \tag{5}$$

$$p = p(\rho_0) \quad \text{in } \Omega\times\{0\}, \tag{6}$$

where ϕ is the porosity of the medium, p the pressure, ρ_w the fluid density, K is a symmetric, uniformly positive definite tensor with components representing the permeability divided by the viscosity, g the magnitude of the gravitational acceleration, D is the depth, and f external mass flow rate; and $\Omega\subset\mathbf{R}^d$, $d=2$ or 3, is the domain and $J=[0,T]$. The equation of state is given by

$$\frac{d\rho_w}{\rho_w} = c_f dp,$$

where c_f is the fluid compressibility. It is useful to introduce a flux variable

$$\mathbf{u} = -K\rho_w(p)(\nabla p - g\rho_w(p)\nabla D).$$

Then we solve for the pressure p and the velocity $\mathbf{u}$ satisfying

$$K^{-1}\rho_w^{-1}(p)\,\mathbf{u} = -\nabla p + g\rho_w(p)\nabla D \qquad \text{in } \Omega\times J, \tag{7}$$

$$\frac{\partial}{\partial t}\phi\rho_w(p) + \nabla\cdot\mathbf{u} = f \qquad \text{in } \Omega\times J, \tag{8}$$

$$p = p_b \quad \text{on } \partial\Omega\times J, \tag{9}$$

$$p = p_0 \quad \text{on } \Omega\times\{0\}. \tag{10}$$

The weak form of (7)–(10) is given by seeking a map $\{\mathbf{u},p,\lambda\}: J\to\mathbf{V}\times W\times M$ such that, for each i,

$$(K^{-1}\rho_w^{-1}(p)\,\mathbf{u},\mathbf{v})_{\Omega_i} = (p,\nabla\cdot\mathbf{v})_{\Omega_i} - \langle\lambda,\mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} \tag{11}$$

$$- \langle p_b,\mathbf{v}\cdot\nu_i\rangle_{\partial\Omega_i\setminus\Gamma} + (g\rho_w(p)\nabla D,\mathbf{v})_{\Omega_i}, \quad \mathbf{v}\in\mathbf{V}_i, \tag{12}$$

$$(\phi\frac{\partial}{\partial t}\rho_w(p),w)_{\Omega_i} + (\nabla\cdot\mathbf{u},w)_{\Omega_i} = (f,w)_{\Omega_i}, \qquad w\in W_i, \tag{13}$$

$$\sum_{j=1}^{n}\langle\mu,\mathbf{u}\cdot\nu_j\rangle_{\Gamma_j} = 0, \qquad \mu\in M, \tag{14}$$

with the initial condition $p=p_0$, where ν_i is the outer unit normal to $\partial\Omega_i$. Note that λ is the pressure on the block interfaces Γ.

Let $\mathcal{T}_{h,i}$ be a conforming, quasi-uniform finite element partition of Ω_i, $1\le i\le n$, of maximal element diameter h_i. Let $h=\max_{1\le i\le n} h_i$. Note that we allow for the possibility that $\mathcal{T}_{h,i}$ and $\mathcal{T}_{h,j}$ need not align on $\Gamma_{i,j}$. Define $\mathcal{T}_h = \cup_{i=1}^n\mathcal{T}_{h,i}$ and let $\mathcal{E}_h$ be the union of all interior edges (faces) not including the interfaces and the outer boundary. Let

$$\mathbf{V}_{h,i}\times W_{h,i}\subset\mathbf{V}_i\times W_i$$

be any of the usual mixed finite element spaces, (e.g., those of Raviart-Thomas-Nedelec, Brezzi-Douglas-Marini). Then let

$$\mathbf{V}_h = \bigoplus_{i=1}^{n}\mathbf{V}_{h,i}, \qquad W_h = \bigoplus_{i=1}^{n} W_{h,i}.$$

Although the normal components of vectors in $\mathbf{V}_h$ are continuous between elements within each block Ω_i, there is no such restriction across Γ.

Let the mortar interface mesh $\mathcal{T}_{H,i,j}$ be a quasi-uniform finite element partition of $\Gamma_{i,j}$ with maximal element diameter $H_{i,j}$. Let $H = \max_{1\le i,j\le n} H_{i,j}$. Define $\mathcal{T}^{\Gamma,H} = \cup_{1\le i<j\le n}\mathcal{T}_{H,i,j}$. Denote by $M_{H,i,j} \subset L^2(\Gamma_{i,j})$ the mortar space on $\Gamma_{i,j}$, containing either the continuous or discontinuous piecewise polynomials of degree m on $\mathcal{T}_{H,i,j}$, where m is at least $k+1$ and k is associated with the degree of the polynomials in $\mathbf{V}_h \cdot \nu$. Now let

$$M_H = \bigoplus_{1\le i<j\le n} M_{H,i,j}$$

be the mortar finite element space on Γ. We require that the following condition be satisfied. For each subdomain Ω_i, define a projection $\mathcal{Q}_{h,i} : L^2(\Gamma_i) \to \mathbf{V}_{h,i}\cdot \nu_i|_{\Gamma_i}$ such that, for any $\phi \in L^2(\Gamma_i)$,

$$\langle \phi - \mathcal{Q}_{h,i}\phi, \mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} = 0, \quad \mathbf{v}\in \mathbf{V}_{h,i}. \tag{15}$$

Assumption 1 *Assume that there exists a constant C, independent of h and H, such that*

$$\|\mu\|_{0,\Gamma_{i,j}} \le C(\|\mathcal{Q}_{h,i}\mu\|_{0,\Gamma_{i,j}} + \|\mathcal{Q}_{h,j}\mu\|_{0,\Gamma_{i,j}}),\ \ \mu\in M_H,\ \ 1\le i<j\le n. \tag{16}$$

Condition (16) says that the mortar space cannot be too rich compared to the normal traces of the subdomain velocity spaces. Therefore in the sequel, we tacitly assume that $h \le H \le 1$. This is not a very restrictive condition, and it is easily satisfied in practice. In the following we treat any function $\mu \in M_H$ as extended by zero on $\partial\Omega$. We remark that $\mathcal{T}_{H,i,j}$ need not be conforming if $M_{H,i,j}$ is discontinuous.

The mixed finite element approximation of (11)–(13) is given by seeking a map $\{\mathbf{u}_h, p_h, \lambda_H\} : J \to \mathbf{V}_h \times W_h \times M_H$ such that, for $1 \le i \le n$,

$$(K^{-1}\rho_w^{-1}(p_h)\,\mathbf{u}_h, \mathbf{v})_{\Omega_i} = (p_h, \nabla\cdot\mathbf{v})_{\Omega_i} - \langle\lambda_H, \mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} \tag{17}$$
$$\quad - \langle p_b, \mathbf{v}\cdot\nu_i\rangle_{\partial\Omega_i\setminus\Gamma} + (g\rho_w(p_h)\nabla D, \mathbf{v})_{\Omega_i} \qquad \mathbf{v}\in\mathbf{V}_{h,i},$$

$$(\phi\frac{\partial}{\partial t}\rho_w(p_h), w)_{\Omega_i} + (\nabla\cdot\mathbf{u}_h, w)_{\Omega_i} = (f, w)_{\Omega_i}, \qquad w\in W_{h,i}, \tag{18}$$

$$\sum_{j=1}^{n}\langle\mu, \mathbf{u}_h\cdot\nu_j\rangle_{\Gamma_j} = 0, \qquad \mu\in M_H, \tag{19}$$

with the initial condition $p_h(0) = \widehat{p_0}$, $L^2(\Omega)$ projection of p_0 onto W_h. It should be noted that within each block Ω_i, we define a standard mixed finite element method, e.g., (18) enforces local conservation on each grid cell. Moreover, $\mathbf{u}_h\cdot\nu$ is continuous on any element face (or edge) $e \not\subset \Gamma\cup\partial\Omega$, and (19) enforces weak continuity of flux across these interfaces with respect to the mortar space M_H.

The unique solvability of the system (17) – (19) follows from condition (16) and the standard argument given in [6].

NUMERICAL RESULTS

In this section we present numerical results for some benchmark problems [5]. In particular we consider two problems: the idealized diagonal channel problem and the fluvial reservoir problem (85th layer of the 10th SPE comparative project), that were previously studied for incompressible flow by Aarnes *et al.* by applying several multiscale mixed methods. Similar solutions were obtained using mortars and they compared favorably. The solutions shown here are of $\log|\mathbf{u}|$ for slightly compressible flow. A fluid compressibility factor of 4.0E-05 was assumed. All computations were run in parallel on up to 16 processors at the beowulf cluster in the Center for Subsurface Modeling at the University of Texas at Austin.

Diagonal Channel

This problem has proved quite challenging to most multiscale methods. Here, a single high-permeability channel goes diagonally from the source to the sink. The domain is a square 64m x 64m x 1m. The permeability is 100 times as high along the diagonal as it is elsewhere. A unit source and unit sink are located at either ends of the high-permeable layer (which is 3 elements thick, away from the boundary). The domain is partitioned into 64 subdomains (8 x 8 coarse mesh). Each subdomain is further sub-divided into an 8 x 8 fine-mesh (giving a fine element size of 1m x 1m).

The result shows the reference solution on the left, on a single-domain (64 x 64 fine mesh) and the mortar solution on the right on an 8 x 8 subdomain partition. Further, we find that by applying a posteriori error estimates loosely based on [8], the mortar degrees of freedom can be chosen to be coarser away from the regions where the error in the solution is higher, while preserving the accuracy of the solution.

Fluvial Reservoir

As a second example, we considered a more realistic fluvial reservoir problem, where the permeability field contains many narrow high flow “channels”. This permeability data was taken from Layer 85 of the 10th SPE Comparative Project [3]. For comparison purposes, although the true measurements are in feet, the solution was calculated in SI units as in [5]. The fine mesh consisted of 60 x 220 elements. Sources and sinks are placed in a five-spot pattern, with a unit source in the middle and sinks of strength 1/4 each at the four corners. The mortar degrees of freedom were again chosen coarse where the errors were small. The errors were highest at the bottom-left and top-right corners of the domain because of the very low permeability, yet equal flow rates at the sinks in these corners.

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