EXTRACTION FINITE ELEMENT METHOD FOR STOKES FLOW WITH SINGULAR FORCES

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ABSTRACT

Recently, the immersed interface finite difference method (IIFDM) of second order accuracy for Navier-Stokes equations developed by Zilin Li. We introduce an immersed interface finite element method(IIFEM) for the Navier-Stokes equations. In this paper, we consider a standard model for the solution of steady incompressible Stokes flow in which a localized force at the interface describes the effect of moving elastic boundaries on a uniform Cartesian grid. The Stokes equations model the flow of a highly viscous fluid with the low Reynolds number and convection terms are dropped from the Navier-Stokes equations.

Our method is designed to handle elastic boundary forces acting at the interface separating the two areas. In each subdomain, the Crouzeix-Raviart(CR) nonconforming elements are used on triangular meshes. In our method, we treat the velocity jump with the IIFEM and eliminate the pressure jump with a new method.

MODEL STOKES PROBLEM

Let $\Omega \in \mathbf{R}^2$ be a convex polygonal domain. The time dependent domain $\Omega(t)$ is separated into two subdomains $\Omega^-(t)$ and $\Omega^+(t)$ with $\bar{\Omega} = \bar{\Omega}^- \cup \bar{\Omega}^+$ and $\Omega^-(t) \cap \Omega^+(t) = \emptyset$. We assume that $\Omega^-(t)$ and $\Omega^+(t)$ are connected and $\partial\Omega^-(t) \cap \partial\Omega^+(t) = \emptyset$. The interface is denoted by $\Gamma(t) = \bar{\Omega}^-(t) \cap \bar{\Omega}^+(t)$. Then, we want to find that the solution $(\mathbf{u}, p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ of the stationary homogeneous Stokes problem for an incompressible viscous fluid confined in Ω satisfies:

$$-\mu\Delta\mathbf{u} + \nabla p = \boldsymbol{g} + \boldsymbol{F}, \quad \text{in } \Omega(t), \tag{1a}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega(t), \tag{1b}$$

$$\mathbf{u} = \mathbf{0}, \quad \text{on } \partial\Omega(t). \tag{1c}$$

with $p = p(\mathbf{x}, t)$ the pressure, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ the velocity, μ the viscosity(the given positive constant) and $\boldsymbol{F}$ the external singular force.

The external singular force can be written as

$$\boldsymbol{F} \equiv (F_1(x,y,t), F_2(x,y,t)) = \int_\Gamma \boldsymbol{f}(s,t)\delta(\mathbf{x} - \mathbf{X}(s,t))\,\mathrm{d}s,$$

where $\mathbf{X}(s,t)$ gives the location of the interface at time t, $\boldsymbol{f}(s,t)$ is the force strength at this point, and δ is the two-dimensional delta function. Note that the time evolution of the flow is governed entirely by the time dependence of the forces $\boldsymbol{F}$. The jumps in the solution result from the fact that the force $\boldsymbol{F}$ is singular and is supported only along the interface $\Gamma(t)$. This singular force leads to the jump of the pressure and the jumps of the first normal derivatives of the pressure and the velocity. We assume continuity of $\mathbf{u}$ across the interface. Now, we introduce the weak formulation of the stationary model is as follows: find $(\mathbf{u}, p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ such that

$$a_\Omega(\mathbf{u}, \mathbf{w}) + b_\Omega(\mathbf{w}, p) \;=\; (\boldsymbol{g}, \mathbf{w}) + \boldsymbol{F}(\mathbf{w}), \qquad {}^\forall \mathbf{w} \in H_0^1(\Omega)^2, \tag{2a}$$

$$b_\Omega(\mathbf{u}, q) \;=\; 0, \qquad {}^\forall q \in L_0^2(\Omega), \tag{2b}$$

where

$$a_{\maltese}(\mathbf{u}, \mathbf{w}) := \int_{\maltese} \mu \nabla \mathbf{u} : \nabla \mathbf{w} \, \mathrm{d}\mathbf{x}, \quad b_{\maltese}(\mathbf{w}, q) = -\int_{\maltese} q \, \mathrm{div} \mathbf{w} \, \mathrm{d}\mathbf{x}$$
$$\boldsymbol{F}(\mathbf{w}) = \int_\Gamma \boldsymbol{f}(s,t) \cdot \mathbf{w}(\mathbf{X}(s,t)) \, \mathrm{d}s$$

JUMP CONDITIONS AND THE EXTRACTION FINITE ELEMENT METHOD

We need to know both the jump in the function and the jump in its normal derivative at each point along the interface.

It is known that the normal velocity must be continuous and the tangential velocity is continuous because of viscosity and a no-slip boundary condition between the elastic band and the fluid on each side. The normal derivative of all variables will, however, be discontinuous in general.

The jump conditions can be written in terms of normal and tangential components of the force $\boldsymbol{f}(s,t)$. We decompose the force as

$$\boldsymbol{f}(s,t) = \begin{bmatrix} f_1(s,t) \\ f_2(s,t) \end{bmatrix} = \begin{bmatrix} \hat{f}_1(s,t)\cos\theta - \hat{f}_2(s,t)\sin\theta \\ \hat{f}_1(s,t)\sin\theta + \hat{f}_2(s,t)\cos\theta \end{bmatrix} \tag{3}$$

where θ is the angle between the x-axis and the normal direction pointing outward from the interface at $\mathbf{X}(s,t)$, and $\hat{f}_1, \ \hat{f}_2$ are the normal and tangential force strengths. So, we have

$$\begin{aligned} \hat{f}_1(s,t) &= f_1(s,t)\cos\theta + f_2(s,t)\sin\theta, \\ \hat{f}_2(s,t) &= -f_1(s,t)\sin\theta + f_2(s,t)\cos\theta. \end{aligned} \tag{4}$$

The jump conditions are the given by

$$[p](s) = \hat{f}_1(s,t), \tag{5a}$$

$$\left[\frac{\partial p}{\partial \mathbf{n}}\right](s) = \frac{\partial \hat{f}_2(s,t)}{\partial s}, \tag{5b}$$

$$\left[\mu \frac{\partial \mathbf{u}}{\partial \mathbf{n}}\right](s) = \hat{f}_2 \mathbf{t} \tag{5c}$$

For each subdomain, we have the following equations :

$$a_{\Omega_\Gamma}(\mathbf{u},\mathbf{w}) + b_{\Omega_\Gamma}(\mathbf{w},p) \;=\; (\boldsymbol{g},\mathbf{w})_{\Omega_\Gamma}, \qquad {}^\forall\mathbf{w}\in\mathbf{V}, \tag{6a}$$

$$b_{\Omega_\Gamma}(\mathbf{u},q) \;=\; 0, \qquad {}^\forall q\in Q, \tag{6b}$$

satisfying jump conditions (5) where $\Omega_\Gamma = \Omega - \Gamma,\ \mathbf{V} = H_0^1(\Omega)^2 \cap H^2(\Omega_\Gamma),\ Q = L_0^2(\Omega)\cap H^2(\Omega_\Gamma)$. Considering the jump conditions, we take

$$p = \hat{p} + p^*, \quad \mathbf{u} = \hat{\mathbf{u}} + \mathbf{u}^* \tag{7}$$

where $\hat{p},\ \hat{\mathbf{u}}$ are continuous and so are their *normal derivatives*, $p^*, \mathbf{u}^*$ satisfy the jump conditions (5). Then, the problem (2) can be written as : find $(\hat{\mathbf{u}},\hat{p})\in\mathbf{V}\times Q$ such that

$$a_{\Omega_\Gamma}(\hat{\mathbf{u}},\mathbf{w}) + b_{\Omega_\Gamma}(\mathbf{w},\hat{p}) \;=\; -a_{\Omega_\Gamma}(\mathbf{u}^*,\mathbf{w}) - b_{\Omega_\Gamma}(\mathbf{w},p^*) + J(\mathbf{w}), \qquad {}^\forall\mathbf{w}\in\mathbf{V}, \tag{8a}$$

$$b_{\Omega_\Gamma}(\hat{\mathbf{u}},q) \;=\; -b_{\Omega_\Gamma}(\mathbf{u}^*,q), \qquad {}^\forall q\in Q, \tag{8b}$$

where

$$J(\mathbf{w}) = \int_\Gamma \left(\left[\mu\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\right] - [p]\right)\mathbf{w}(\mathbf{X}(s,t))\cdot\mathbf{n}\,\mathrm{d}s, \quad [\,\cdot\,]_\Gamma = \lim_{\mathbf{x}\to\Gamma}(\,\cdot\,|_{\Omega^-(\mathbf{x})} - \,\cdot\,|_{\Omega^+(\mathbf{x})})$$

$$\begin{pmatrix} [\hat{p}](s) = 0 & [\hat{\mathbf{u}}](s) = \mathbf{0} \\ \left[\dfrac{\partial\hat{p}}{\partial\mathbf{n}}\right](s) = 0 & \left[\mu\dfrac{\partial\hat{\mathbf{u}}}{\partial\mathbf{n}}\right](s) = \mathbf{0} \end{pmatrix} \tag{9}$$

$$\begin{pmatrix} [p^*](s) = \hat{f}_1(s,t) & [\mathbf{u}^*](s) = \mathbf{0} \\ \left[\dfrac{\partial p^*}{\partial\mathbf{n}}\right](s) = \dfrac{\partial\hat{f}_2(s,t)}{\partial s} & \left[\mu\dfrac{\partial\mathbf{u}^*}{\partial\mathbf{n}}\right](s) = \hat{f}_2\mathbf{t} \end{pmatrix} \tag{10}$$

Now, we describe the immersed interface finite element space. Let $\{\mathcal{T}_h\}$ be the usual consistent(i.e. no hanging nodes) shape regular finite element triangulations of the domain Ω. We call an element $T\in\mathcal{T}_h$ an interface element if the interface Γ passes through the interior of T, otherwise we call T a noninterface element. Let $\mathcal{T}_h^I$ be the collection of all interface elements and let $\mathcal{T}_h^N$ be the collection of all interface elements.

As usual, we want to construct local basis functions on each element T of the partition $\mathcal{T}_h$. Let $\mathbf{V}_h, Q_h$ be a discrete space pare of finite element spaces of $\mathbf{V}, Q$. Then, we assume that for each $\mathbf{v}_h\in\mathbf{V}_h,\ q_h\in Q_h$,

$$\mathbf{v}_h = \hat{\mathbf{v}}_h + \mathbf{v}_h^*, \qquad q_h = \hat{q}_h + q_h^* \tag{11}$$

where

$$\hat{\mathbf{v}}_h \in \hat{\mathbf{V}}_h := \left\{\mathbf{v}\in\mathcal{P}_1(T)^2 |\ T\in\mathcal{T}_h^N\right\},$$

$$\mathbf{v}_h^* \in \mathbf{V}_h^* := \left\{\mathbf{v}\in\mathcal{P}_1(T)^2 |\ T\in\mathcal{T}_h^I\cap \maltese,\ \maltese = \Omega^+(t),\Omega^-(t)\right\},$$

$$\hat{q}_h \in \hat{Q}_h := \left\{q\in\mathcal{P}_0(T) |\ T\in\mathcal{T}_h^N\right\},$$

$$q_h^* \in Q_h^* := \{q\in\mathcal{P}_1(T) |\ T\in\Omega^-(t)\}.$$

and $\hat{()}, ()^*$ satisfy the jump conditions (9), (10), respectively. Now, the discretization of (8) is as follows: determine $(\hat{\mathbf{u}}_h, \hat{p}_h) \in \hat{\mathbf{V}}_h \times \hat{Q}_h$ such that

$$a_{\Omega_\Gamma}(\hat{\mathbf{u}}_h, \mathbf{w}_h) + b_{\Omega_\Gamma}(\mathbf{w}_h, \hat{p}_h) = -a_{\Omega_\Gamma}(\mathbf{u}_h^*, \mathbf{w}_h) - b_{\Omega_\Gamma}(\mathbf{w}_h, p_h^*) + J(\mathbf{w}_h), \qquad {}^\forall \mathbf{w}_h \in \hat{\mathbf{V}}_h, \tag{12a}$$

$$b_{\Omega_\Gamma}(\hat{\mathbf{u}}_h, q_h) = -b_{\Omega_\Gamma}(\mathbf{u}_h^*, q_h), \qquad {}^\forall q_h \in \hat{Q}_h, \tag{12b}$$

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