Non-Uniform Fast Fourier Transformation (NUFFT) and Magnetic Resonance Imaging (MRI)

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ABSTRACT

The type 1, type 2, and type 3 nonuniform fast Fourier transformation(NUFFT) and fast sinc transformation have been numerically implemented. Computational examples and numerical algorithms are provided in order to demonstrate computational efficiency and feasibility of the tools for magnetic resonance imaging (MRI) reconstruction applications.

1. THE NON-UNIFORM FFT (NUFFT)

We start with the definition of nonuniform fast Fourier transformation.

1. **Type1-NUFFT** : The nonuniform discrete Fourier transform of types 1

$$F(k_1,k_2) = \frac{1}{N}\sum_{j=0}^{N-1} f_j\, e^{-i(k_1,k_2)\cdot\mathbf{x}_j} \ , \tag{1}$$

It is convenient to think of (1) as a discretization of the Fourier integral

$$F(k_1,k_2) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f(\mathbf{x})\, e^{-i(k_1,k_2)\cdot\mathbf{x}}\, d\mathbf{x} \tag{2}$$

with $\{\mathbf{x}_j\}$ serving as the discretization points. If we let w_j denote the quadrature weight corresponding to $\{\mathbf{x}_j\}$, then we obtain (1) by setting $f_j = f(\mathbf{x}_j)\, w_j$.

2. **Type2-NUFFT** : The nonuniform discrete Fourier transform of types 2

$$f(\mathbf{x}_j) = \sum_{k_1}\sum_{k_2} F(k_1,k_2)\, e^{i(k_1,k_2)\cdot\mathbf{x}_j}, \tag{3}$$

where $\mathbf{x}_j \in [0,2\pi]\times[0,2\pi]$ and $-\frac{M}{2} \le k_1, k_2 < \frac{M}{2}$. Equation (3), of course, is simply the evaluation of a finite Fourier series

$$f(\mathbf{x}) = \sum_{k_1}\sum_{k_2} F(k_1,k_2)\, e^{i(k_1,k_2)\cdot\mathbf{x}} \tag{4}$$

at an arbitrary set of targets.

3. **Type3-NUFFT** : The nonuniform discrete Fourier transform of types 3

$$F_k = \sum_{j=0}^{N-1} f_j\, e^{\pm i\mathbf{s}_k\cdot\mathbf{x}_j} \ , \tag{5}$$

at N locations $\mathbf{s}_k$. We can think of (5) as a discretization of the continuous Fourier transform,

$$F(\mathbf{s}) = \frac{1}{(2\pi)^d} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(\mathbf{x})\, e^{\pm i\mathbf{s}\cdot\mathbf{x}}\, d\mathbf{x} \tag{6}$$

using nonuniformly sampled discretization points and evaluated at nonuniformly sampled frequencies.

2. MAGNETIC RESONANCE IMAGING (MRI)

The MRI hardware is able to acquire the Fourier transform of proton density of a specimen such as human body at selected points in the frequency domain. Under the assumption of a perfectly homogeneous magnetic field, the signal produced at time t is given by

$$s(t) = \int \rho(\mathbf{x}) e^{-i2\pi \mathbf{k}(t)\cdot\mathbf{x}}\, d\mathbf{x}. \tag{7}$$

In other words, $s(t)$ is precisely the value of the Fourier transform $\tilde{\rho}$ at the location $\mathbf{k}(t) = (k^1(t), k^2(t))$. In most clinical systems, the device is designed to acquire data $\tilde{\rho}(k^1_{m_1}, k^2_{m_2})$ on a uniform Cartesian mesh, from which a standard FFT can be used for image reconstruction.

$$\rho(x^1_{n_1}, x^2_{n_2}) = C \sum_{m_1=0}^{M-1} \sum_{m_2=0}^{M-1} e^{2\pi i \frac{m_1 n_1}{M}} e^{2\pi i \frac{m_2 n_2}{M}} \tilde{\rho}(k^1_{m_1}, k^2_{m_2})$$

For a variety of technical reasons, however, nonuniform data sampling techniques are much better suited for fast data acquisition, motion correction, and functional MRI [1,7]. In order to demonstrate a fast MRI reconstruction method using a NUFFT, we create simulated MRI data by using a type 2 transformation in two dimensions:

$$F(s^1_k, s^2_k) = \sum_{j_1} \sum_{j_2} f(j_1, j_2)\, e^{-i(j_1,j_2)\cdot(s^1_k, s^2_k)}\,,$$

followed by a type 1 transformation to reconstruct the image,

$$\tilde{f}(j_1, j_2) = \sum_{k=0}^{N-1} F_k\, e^{i(j_1,j_2)\cdot(s^1_k, s^2_k)}. \tag{8}$$

Once the decision has been made to use (8) for reconstruction, one still has a number of degrees of freedom to work with. Engineering considerations determine the selection of points $\{(s^1_k, s^2_k)\}$ which will certainly affect the image quality. One must also select quadrature weights W_k so that, in the transform (8), $F_k \equiv W_k\, F(s^1_k, s^2_k)$. The values $\{W_k\}$ can be considered quadrature weights, and it is shown in [2,5] that an optimal set of weights is given by the formula

$$\frac{1}{W_k} = \sum_m \mathrm{sinc}^2((s^1_m, s^2_m) - (s^1_k, s^2_k)). \tag{9}$$

Here, $\mathrm{sinc}(k) \equiv \frac{\sin(\pi k)}{\pi k}$ and, in d dimensions, we define $\mathrm{sinc}(\mathbf{k}) = \mathrm{sinc}(k_1) \cdot \mathrm{sinc}(k_2) \cdots \mathrm{sinc}(k_d)$, where $\mathbf{k} = (k_1, k_2, \dots, k_d)$.

Following the discussion of [8], the minimum-norm least-squares solution to this problem, denoted by $\hat{\rho}(x)$, can be found by applying the pseudo-inverse of the operator $\mathcal{H}$ to the signal,

$$\hat{\rho}(x) = \mathcal{H}^{+}\mathbf{s} = \mathcal{H}^{\dagger}(\mathcal{H}\mathcal{H}^{\dagger})^{+}\mathbf{s}, \tag{10}$$

where $\mathcal{H}^\dagger$ is the adjoint of $\mathcal{H}$ and $(\mathcal{H}\mathcal{H}^\dagger)^+$ is the pseudoinverse of $\mathcal{H}\mathcal{H}^\dagger$. Note that the process involved in (8) can be written as

$$\rho(\mathbf{r}) \approx \mathcal{H}^\dagger W \mathbf{s}$$

where W is the diagonal matrix of quadrature weights. Thus, the quadrature approach based on the inverse Fourier transform can be viewed as a diagonal approximation ($W\mathbf{s}$) of the pseudoinverse construction ($(\mathcal{H}\mathcal{H}^\dagger)^+\mathbf{s}$).

3. CONCLUSIONS

- We have presented a simple version of the type 1, type 2 , and type 3 nonuniform FFT. We observe that one of the standard interpolation or "gridding" schemes, based on Gaussians, can be accelerated by a significant factor without precomputation and storage of the interpolation weights. This is of particular value in two and three dimensional settings, saving either $10^d N$ in storage in d dimensions or a factor of about $5-10$ in CPU time (independent of dimension).
- Type 3 NUFFT can be used to approximate the continuous Fourier transform when neither the spatial nor the Fourier domain spacing is regular. For examples, it allows the evaluation of MRI signal in the presence of a (field inhomogeneity given by $\phi(\mathbf{x})$,

$$s(t) = \int \rho(\mathbf{x}) e^{-i2\pi\mathbf{k}(t)\cdot\mathbf{x}} e^{-i\phi(\mathbf{x})t}\, d\mathbf{x}. \tag{11}$$

  which requires the computation of

$$s(t_j) \approx \sum_{n=1}^{N} w_n \rho(\mathbf{x}_n) e^{-i2\pi\mathbf{k}(t_j)\cdot\mathbf{x}_n} e^{-i\phi(\mathbf{x}_n)t_j} = \sum_{n=1}^{N} w_n \rho(\mathbf{x}_n) e^{-i\mathbf{K}_j\cdot\mathbf{X}_n}$$

  where $\mathbf{K}_j = (k_1(t_j), k_2(t_j), t_j)$ and $\mathbf{X}_n = (2\pi x_n^1, 2\pi x_n^2, \phi(\mathbf{x}_n))$.
- We have constructed a fast algorithm for the (discrete) sinc and sinc^2 transforms which have immediate application in MR image reconstruction. These two algorithms will also accelerate, for example, the band-limited interpolation method of [2].
- We expect that the algorithms described here will be of fairly broad utility since non-uniform fast Fourier transformation and sinc convolution arises naturally in many signal and image processing contexts.

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