MONTE: THE METHOD OF NONFLAT TIME EVOLUTION IN PDE-BASED IMAGE RESTORATION

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ABSTRACT

This article is concerned with effective numerical techniques for partial differential equation (PDE)-based image restoration. Numerical realizations of most PDE-based denoising models show a common drawback: loss of fine structures. In order to overcome the drawback, the article introduces a new time-stepping procedure, called the *method of nonflat time evolution* (MONTE), in which the timestep size is determined based on local image characteristics such as the curvature or the diffusion magnitude. The MONTE provides the PDE-based restoration models with an effective mechanism for the equalization of the net diffusion over a wide range of image frequency components. It can be easily applied to diverse evolutionary PDE-based restoration models and their spatial and temporal discretizations. It has been numerically verified that the MONTE results in a significant reduction in nonphysical dissipation and preserves fine structures such as edges and textures satisfactorily, while it removes the noise with an improved efficiency. Various numerical results are shown to confirm the claim.

A GENERAL DENOISING MODEL

Let f be the observed image of the form

$$f = u + v, \tag{1}$$

where u is a desired image and v denotes noise or the residual. Then, most PDE-based image denoising models can be formulated in their evolutionary form as

$$\frac{\partial u}{\partial t} + \mathcal{L}u \,=\, \mathcal{R}(f-u), \tag{2}$$

where $\mathcal{L}$ is a nonlinear diffusion operator and $\mathcal{R}$ denotes a nonnegative constraint term. For example, the Perona-Malik (PM) model [5], the motion by mean curvature (MMC), the TV model [6], the improved total variation (ITV) model [4], and the $\alpha\,\beta\,\omega$ (ABO) model [3] can be expressed as (2) with the following choices:

$$
\begin{aligned}
&\text{(a) } \mathcal{L}u = -\nabla\cdot\left(g(|\nabla u|)\,\nabla u\right), && \mathcal{R}=0, && \text{(PM)}\\
&\text{(b) } \mathcal{L}u = -|\nabla u|\,\kappa_1(u), && \mathcal{R}=0, && \text{(MMC)}\\
&\text{(c) } \mathcal{L}u = -\kappa_1(u), && \mathcal{R}=\lambda, && \text{(TV)}\\
&\text{(d) } \mathcal{L}u = -|\nabla u|\,\kappa_1(u), && \mathcal{R}=\lambda\,|\nabla u|, && \text{(ITV)}\\
&\text{(e) } \mathcal{L}u = -|\nabla u|^{\alpha}\,\kappa_\omega(u), && \mathcal{R}=\beta(\mathbf{x},t), && \text{(ABO)}
\end{aligned}
\tag{3}
$$

where $\lambda > 0$, $\alpha \geq 0$, $\omega \geq 1$ are constants and $\kappa_q(u)$ and g are functions of the form

$$
\begin{aligned}
\kappa_q(u) &= \nabla\cdot\left(\frac{\nabla u}{|\nabla u|^q}\right), \quad q\geq 0,\\
g(s) &= 1/(1+s^2/\sigma^2), \quad \sigma>0.
\end{aligned}
\tag{4}
$$

Here β is a nonnegative function of f and u; see [3] for an automatic and yet effective strategy of choosing a variable constraint parameter. The ABO model is a generalization of the ITV model, incorporating cases of non-convex minimization ($\omega > 1$) and a more reliable constraint parameter. It has been numerically verified that for most real images, the ABO model performs better than the ITV model for $1 < \alpha = \omega < 2$ and the best for $\alpha = \omega = 1.5 \sim 1.9$ [3]; see also numerical results in Section below. The ABO model has been applied as an effective edge-forming method for image zooming which incorporates large and non-integer magnification factors [1,2].

THE NUMERICAL ALGORITHM

For a numerical discretization, we let Δt be a timestep size and let $t^n = n\Delta t$ and $u^n = u(\cdot, t^n)$. Then, an *incomplete* (linearized) Crank-Nicolson scheme for the general model (2) reads

$$
\frac{u^n - u^{n-1}}{\Delta t} + \mathcal{A}^{n-1}\,\frac{u^n + u^{n-1}}{2} = \mathcal{R}^{n-1}\,f, \tag{5}
$$

where $\mathcal{A}^{n-1}$ is the diffusion matrix which can be defined depending on the selected model. For example, for the ABO model, one can set $\mathcal{A}^{n-1} = \mathcal{A}_1^{n-1} + \mathcal{A}_2^{n-1}$, where

$$
\mathcal{A}_\ell^{n-1}\,u^m := -|\nabla_h u^{n-1}|^{\alpha}\,D_{x_\ell}\Big(\frac{D_{x_\ell}u^m}{|\nabla_h u^{n-1}|^{\omega}}\Big) + \frac{\mathcal{R}^{n-1}}{2}u^m, \tag{6}
$$

for $\ell = 1,\, 2$ and $m = n,\, n-1$. Here $(D_{x_1}, D_{x_2})^T$ is the half-step central difference operator for the gradient ∇ and $|\nabla_h u^{n-1}|$ denotes a numerical approximation of gradient magnitude $|\nabla u|$ such as the (standard) second-order central scheme. The linear system (5) may be solved by directly applying an iterative algebraic solver. However, it can be perturbed in order to apply the *alternating direction implicit* (ADI) time-stepping procedure:

$$\begin{aligned}\Big(1+\frac{\Delta t}{2}\mathcal{A}_1^{n-1}\Big)u^* &= \Big(1-\frac{\Delta t}{2}\mathcal{A}_1^{n-1}-\Delta t\mathcal{A}_2^{n-1}\Big)u^{n-1}\\ &\quad +\Delta t\,\mathcal{R}^{n-1}\,f,\\ \Big(1+\frac{\Delta t}{2}\mathcal{A}_2^{n-1}\Big)u^n &= u^*+\frac{\Delta t}{2}\mathcal{A}_2^{n-1}u^{n-1},\end{aligned} \tag{7}$$

where u^* is an intermediate solution. In this article we will call (7) the *Crank-Nicolson ADI* (CN-ADI) algorithm.

THE MONTE TIME-STEPPING PROCEDURE

In the simulation of the model (2) utilizing the CN-ADI (7), we try to control the amount of diffusion by setting the timestep size Δt dynamically, depending on the diffusion magnitude for the last iterate $|\mathcal{L}^{n-1}u^{n-1}|$. (Note that u^n is not available at the beginning of the n-th time level.) That is, we set $\Delta t^n = \Delta t(\cdot, t^n)$ as

$$\Delta t(\mathbf{x}, t^n) = \Delta t_0 \cdot F(\mathcal{L}u(\mathbf{x}, t^{n-1})), \tag{8}$$

where Δt_0 is a constant and F is a scaling function. For example, we may define F as follows: for positive constants η and γ,

$$F(s) = \frac{\gamma}{1+\eta\,|s|}. \tag{9}$$

Incorporating (8) into (7), the resulting algorithm reads

$$\begin{aligned}\Big(1+\frac{\Delta t^n}{2}\mathcal{A}_1^{n-1}\Big)u^* &= \Big(1-\frac{\Delta t^n}{2}\mathcal{A}_1^{n-1}-\Delta t^n\mathcal{A}_2^{n-1}\Big)u^{n-1}\\ &\quad +\Delta t^n\,\mathcal{R}^{n-1}f,\\ \Big(1+\frac{\Delta t^n}{2}\mathcal{A}_2^{n-1}\Big)u^n &= u^*+\frac{\Delta t^n}{2}\mathcal{A}_2^{n-1}u^{n-1}.\end{aligned} \tag{10}$$

The numerical solution u^n in (10) has values on different time levels; it is defined on a nonflat time surface. The algorithm seeks the solution through a nonflat time evolution, for which we call the strategy the *method of nonflat time evolution* (MONTE). We will call (10) the *MONTE-incorporated CN-ADI* (M-CN-ADI) algorithm.

NUMERICAL EXPERIMENTS

In Figure 1, we choose (a) Lenna in gray-scale as a trial image, and (b) the image is perturbed by a Gaussian noise of PSNR=24.8 .

We have implemented four different models: ITV, ABO, the ITV incorporating the MONTE (M-ITV), and the ABO incorporating the MONTE (M-ABO). Figures from (c) to (f) depict restored images from the noisy image in (b) by the four models. As one can see from (c), the ITV model introduces a large nonphysical dissipation for the restored image to be blurry. The ABO and M-ITV models lose less fine details in their resulting images, while the M-ABO model shows the best restored image as in (f).

Figure 1. Lenna: (a) The original image, (b) a noisy image contaminated by a Gaussian noise (PSNR=24.8). (c) $\sim$ (f): Restored images from (b) using (c) ITV (d), ABO (e), M-ITV, and (f) M-ABO.

We have tested lots of other images as well, and it has been numerically verified that the MONTE is efficient and reliable for both synthetic and real images. The new technique has shown attractive characteristics such as: (a) significant reduction of nonphysical dissipation, (b) successful preservation of fine structures, (c) convenient applicability to diverse PDE-based restoration models, and (d) improved efficiency.

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