Accurate Simulation of Transient Multiple Scattering

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ABSTRACT

Starting from a high-order local nonreflecting boundary condition(NBC) for single scattering [6], we derive a local NBC for time-dependent multiple scattering problems, which is completely *local both in space and time*. To do so, we first develop a high order exterior evaluation formula for a purely outgoing wave field, given its values and those of certain auxiliary functions needed for the local NBC on the artificial boundary. By combining that evaluation formula with the decomposition of the total scattered field into purely outgoing contributions, we obtain the first exact, completely local, NBC for time-dependent multiple scattering. Remarkably, the information transfer (of time retarded values) between sub-domains will only occur across those parts of the artificial boundary, where outgoing rays intersect neighboring sub-domains, i.e. typically only across a fraction of the artificial boundary. The accuracy, stability and efficiency of this new local NBC is evaluated by coupling it to standard finite element or finite difference methods.

1 LOCAL NBC FOR SINGLE SCATTERING

We wish to calculate numerically the time dependent field u scattered from a bounded scattering region in three-dimensional space. In this region, there may be one or more scatterers, and the equation for u may have variable coefficients and source terms. As usual, we surround the scattering region by an artificial boundary B, and confine the computation to the region Ω bounded by B. Then, a nonreflecting boundary condition (NBC) is needed at B, which guarantees that the solution of the problem in Ω coincides with the solution of the original problem in the unbounded region.

We let B be the sphere of radius R and assume that u satisfies the homogeneous wave equation,

$$u_{tt} - c^2 \Delta u = 0 \tag{1}$$

outside B. Then, Hagstrom and Hariharan [6] derived the following exact local NBC in three space dimensions:

$$\begin{gathered}\Big(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{1}{r}\Big)u = w_1, \\ \Big(\frac{1}{c}\frac{\partial}{\partial t} + \frac{k}{r}\Big)w_k = \frac{1}{4R^2}\Big(k(k-1) + \Delta_S\Big)w_{k-1} + w_{k+1}\end{gathered} \tag{2}$$

for $k = 1, 2, \ldots$, and $w_0 = 2u$. Here, Δ_S denotes the Laplace-Beltrami operator in spherical coordinates (r, θ, ϕ). The boundary condition (2) is *local in space and time* and does not involve high-order derivatives, but instead an infinite sequence of auxiliary variables w_k defined on B. Then, the boundary condition remains exact for any combination of spherical harmonics up to order P, while the error introduced at B generally behaves like R^{-2P-1}. Hence, P can always be chosen large enough to reduce the error introduced at B below the discretization error inside the computational domain, at any fixed R. Because it does not involve high-order derivatives, this local boundary condition is easily combined with standard numerical methods and enables arbitrarily high order implementations. Recently, it was extended to the time dependent Maxwell equations [2].

2 EXTERIOR EVALUATION FORMULA

When the solution consists of a finite sum of spherical harmonics up to order P, the local NBC (2) with $k = 0, \ldots, P$ becomes exact. Then, the past values of u and the auxiliary functions w_k at $r = R$ determine the solution *everywhere outside* Ω through the following exact (analytical) representation [5]:

$$u^{[P]}(r, \theta, \phi, t) = \frac{R}{r} \sum_{k=0}^{P} \frac{2^{k-1}}{k!} \left(R \left(1 - \frac{R}{r} \right) \right)^k w_k(R, \theta, \phi, t - \tfrac{r-R}{c}). \tag{3}$$

For a general wave field, equation (3) yields an approximate evaluation formula for u in the exterior region, whose accuracy improves with increasing P (or R).

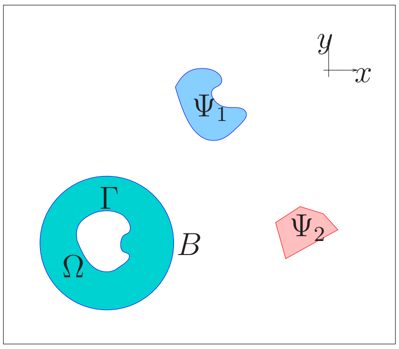

Figure 1. Wave scattering from an obstacle Γ. The computational domain, Ω, is bounded by the artificial boundary B, where the local NBC (2) is imposed. Subsequent evaluation of the solution in other sub-domains, $\Psi_{1,2}$ is possible via (3) by using past values of u and w_k at B.

3 LOCAL NBC FOR MULTIPLE SCATTERING

For simplicity, we consider a scattering problem with two bounded disjoint scatterers, each surrounded by a sphere B_i of radius, R_i $i = 1, 2$. Hence, the entire artificial boundary $B = B_1 \cup B_2$ and the computational domain $\Omega = \Omega_1 \cup \Omega_2$. In contrast to the situation of single scattering above, we cannot simply expand u outside B as a superposition of purely outgoing wave fields. In fact, since part of the scattered field leaving Ω_1 will reenter Ω_2 at later times, and vice versa, u is not outgoing ouside Ω. Thus, the boundary condition we seek at B must not only let outgoing waves leave Ω_1 without spurious reflection from B_1, but also propagate those

waves to Ω_2, and so forth, without introducing any spurious reflections.
Following [4], we first decompose the scattered field u in two wave fields, $u = u_1 + u_2$, where u_i is purely outgoing as seen from Ω_i. The two wave fields u_1 and u_2 both solve the homogeneous wave equation (1) outside Ω, and their sum coincides with u. The outgoing field u_1^{out}, as seen from Ω_1, is fully determined by its boundary values on B_1, while the incoming field u_{12}^{in} from Ω_2 to Ω_1 is fully determined by its boundary values on B_2. Next, we apply $c^{-1}\partial_t + \partial_{r_i} + R_i^{-1}$ in local spherical coordinates (r_i, θ_i, ϕ_i) to u on each artificial boundary component B_i, $i = 1, 2$. This yields the following exact local NBC for multiple scattering:

$$\begin{aligned} \mathcal{B}_1 u|_{B_1} &= \left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial r_1} + \frac{1}{R_1}\right) u|_{B_1} & \mathcal{B}_2 u|_{B_2} &= \left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial r_2} + \frac{1}{R_2}\right) u|_{B_2} \\ &= \mathcal{B}_1 u_1^{out} + \mathcal{B}_1 u_{12}^{in} \quad \text{on } B_1, & &= \mathcal{B}_2 u_2^{out} + \mathcal{B}_2 u_{21}^{in} \quad \text{on } B_2. \end{aligned} \tag{4}$$

To evaluate $\mathcal{B}_1 u_1^{out}$ we use (2) at B_1, whereas to evaluate $\mathcal{B}_1 u_{12}^{in}$ we use (3) for u_2 on B_1. The needed past values of w_k are stored on each B_i at regular time and angular intervals and calculated, as needed, via local spline interpolation [1]. Because those values are time-retarded, they are already known, so that the entire scheme remains explicit in time. Remarkably, the information transfer (of time retarded values) between sub-domains occurs only across those parts of the artificial boundary, where outgoing rays intersect neighboring sub-domains, i.e. typically only across a fraction of the artificial boundary.

4 COMPUTATIONAL RESULTS

To demonstrate the accuracy of the local NBC combined with the exterior evaluation formula (3), we first consider the following simple problem, for which the exact solution is an outgoing spherical wave generated by a Gaussian point source located at distance $d = 0.4$ from the origin. The exact solution is used to initialize the numerical solution inside the computational domain $\Omega = \{(r, \theta) \,|\, r \in [0.5, 1], \theta \in [0, \pi])\}$ and we impose (2) for varying P on the artificial boudary located at $R = 1$.
In Fig. 2 we show the L^2-error inside Ω vs. the mesh size h at the final time $t = 0.6$. For $P = 4$ we observe the expected global second-order convergence up to the finest mesh chosen here; further mesh refinement generally requires increasing the value of P.

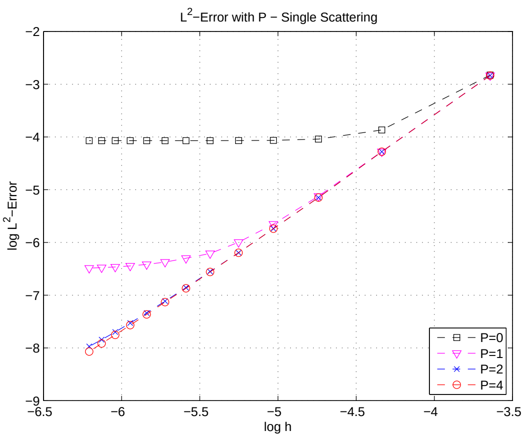

Figure 2. The L^2-error in a single computational domain

Next, we demonstrate the accuracy and performance of our local NBC (4) in a situation of multiple scattering. A plane wave generated between two spherical obstacles initially propagates to the right. It then impinges on the right sphere and bounces back and forth between the two obstacles without spurious reflections, as shown in Fig. 3. Here the computational domain consists of two disjoint regions, each surrounding an inner spherical obstacle. This test problem is axisymmetric in a spherical coordinate, so that the computation is restricted to two dimensions.

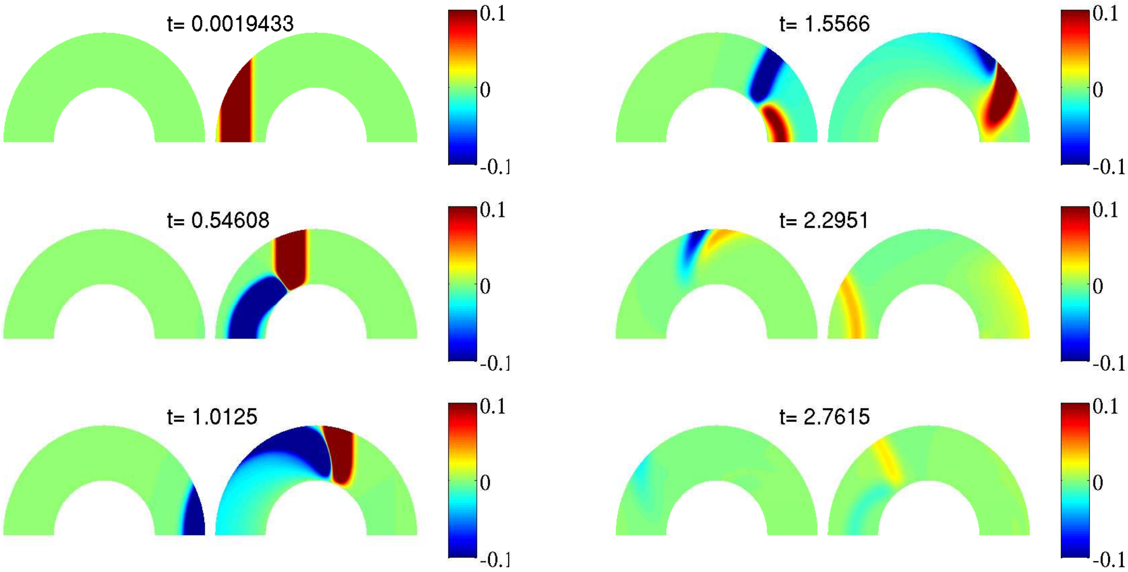

Figure 3. Plane wave scattering from two sound-soft spheres. The computation is restricted to the two disjoint regions.

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