Constraint-Preserving Numerical Methods for Hyperbolic Partial Differential Equations

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ABSTRACT

New numerical methods are introduced for hyperbolic partial differential equations with intrinsic constraints. Such equations occur frequently in physics and engineering. The constraints, though satisfied on the analytical level, are difficult to control in a discrete setting. The new numerical methods are exactly constraint-preserving in the sense that the local value of a specific discrete constraint operator is exactly preserved during the evolution.

1 INTRINSIC CONSTRAINTS

A number of hyperbolic conservation laws have intrinsic constraints like vanishing divergences or constant curl. These constraints look like elliptic or global operators but do not change the character of the underlying conservation law, i.e., the equations remain hyperbolic with finite speed of propagation. Popular examples are the wave equation system, Maxwell's equations, the equations of magnetohydrodynamics and Einstein equations. For instance, the evolution equation of the magnetic field and its constraint in magnetohydrodynamics have the form

$$\partial_t \mathbf{B} + \operatorname{div}\left(\mathbf{B}\,\mathbf{v}^T - \mathbf{v}\,\mathbf{B}^T\right) = 0 \quad \Rightarrow \quad \operatorname{div}\mathbf{B} = const \text{ in time,} \tag{1}$$

similarly, the wave equation system

$$\begin{aligned} \partial_t \rho + \operatorname{div}\mathbf{m} &= 0 \\ \partial_t \mathbf{m} + \operatorname{div}\rho\mathbf{I} &= 0 \end{aligned} \quad \Rightarrow \quad \operatorname{curl}\mathbf{m} = const \text{ in time} \tag{2}$$

has a curl-constraint. These constraints are called intrinsic, since they follow solely from the evolution equation and do not form an independent equation. In addition, the existence of the constraints do not change the mathematical character of the system.

2 NUMERICAL METHODS

The examples of equations given above form hyperbolic conservation laws. With the finite speed of propagation finite volume methods are the proper choice for numerical method. Unfortunately, most of the common implementations do not honor the constraint and numerical results suffer from strong errors in the constraint preservation.

Guided by multidimensional discretizations of hyperbolic conservation laws this talk introduces a framework how to incorporate the constraints into the interface flux formulation of a numerical finite volume method, see [1]. This approach keeps the well known advantages of finite

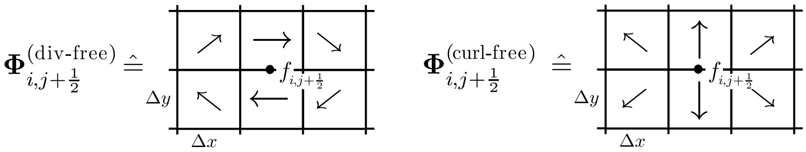

Figure 1. Special shape of div-preserving and curl-preserving flux distributions.

volume methods like shock capturing and upwinding while introducing an exact preservation of a constraint discretization. The key quantity in this theory of constraint-preserving schemes is the so-called *flux distribution*, $\mathbf{\Phi}_{\hat{K}}$, a locally supported grid function which describes how the flux emerging from cell $\hat{K}$ is distributed in the grid. Any finite volume scheme for a vector field $\mathbf{u}$ can be written in the form

$$\tilde{\mathbf{u}}^{m+1}\Big|_K = \tilde{\mathbf{u}}^m\big|_K + \sum_{\hat{K}\in\{K\}\cup\mathcal{N}(K)} \mathbf{\Phi}_{\hat{K}}\left(\tilde{\mathbf{u}}^m\right)\big|_K . \tag{3}$$

based on flux distributions. Constraint preservation is now specified in terms of the flux distributions. Once special conditions on the shape of the flux distribution are satisfied, see Fig. 1, the resulting method will be exactly constraint-preserving.

The ideas are derived and tested by means of linear, multidimensional advection equations with constraints. It turns out that these equations play a similar role for the design of constrained fluxes as the scalar advection equation does for the design of non-constrained fluxes. Applications of the framework and numerical results will be shown for the case of constrained divergence-free and curl-free advection, magnetohydrodynamics, see [2], and the wave equation system, see Fig. 2.

REFERENCES

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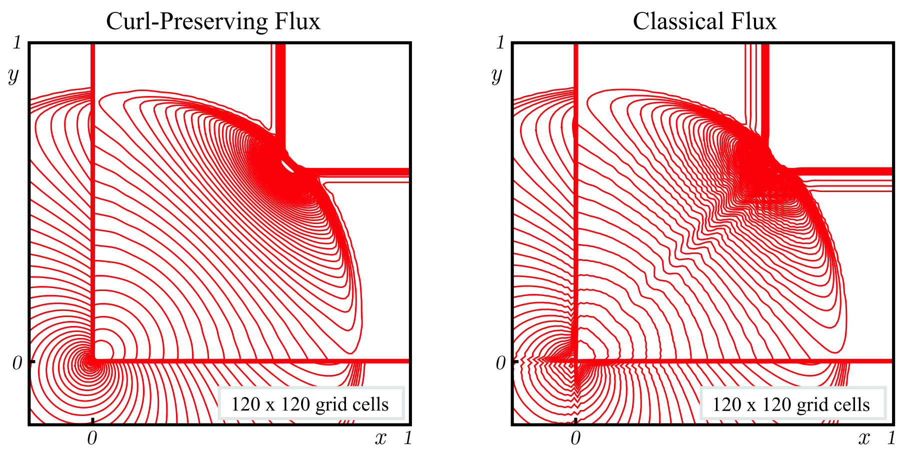

Figure 2. 2D-Riemann problem for the wave equation system. The contour plots show $\|\mathbf{m}\|$ for a standard (right) and the new curl-preserving method (left). The standard method shows asymmetry and spurious oscillations.