On Lower Semicontinuity of the KKT Point Set in Quadratic Programs under Linear Pertubations

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ABSTRACT

The problem of minimizing or maximizing a linear-quadratic function on a convex polyhedral set is called a quadratic program. Since the appearance of the paper by Daniel (Ref. 1) in 1973, continuity and differentiablity properties of the solution map, the local solution map, the Karush-Kuhn-Tucker (KKT, for brevity) point set mapping and the optimal value function in parametric quadratic programming have been studied widely in the literature. In particular, upper semicontinuity and also lower semicontinuity of the KKT point set mapping in indefinite quadratic programs under perturbations has been investigated in Refs. 2–4. In Refs. 2–4, it was assumed that every component of the data is subject to perturbation. If only the linear part of the data is subject to perturbation, then the upper semicontinuity of the KKT point set mapping can be studied via a theorem of Robinson (Ref. 5) on the upper Lipschitz continuity of polyhedral multifunctions.

The aim of this paper is derive necessary and sufficient conditions for the lower semicontinuity of the Karush-Kuhn-Tucker point set in indefinite quadratic programs under linear perturbations. The necessary conditions are relatively simple. But the sufficient conditions are rather sophisticated. A series of examples is designed to show how each set of the sufficient conditions can be realized in practice.

We consider the following indefinite quadratic program

$$\text{Minimize} \quad f(x) := \frac{1}{2}x^T Dx + c^T x \quad \text{subject to } \ x \in \Delta(A, b), \tag{1}$$

where $\Delta(A, b) = \{x \in R^n : Ax \geq b\}$, D is a symmetric $(n \times n)-$matrix, A is an $(m \times n)-$matrix, $b \in R^m$ and $c \in R^n$ and some given vectors. Here the superscript ^{T} denotes transposition. In what follows, matrices D and A will be fixed, while vectors c and b are subject to change. Since D is not assumed to be a positive semidefinite matrix, function f is not necessarily convex. Thus we will have deal with indefinite quadratic programs under linear perturbations.

We say that $x \in R^n$ is a Karush-Kuhn-Tucker point of (1) if there exists a Lagrange multi-

plier $\lambda \in R^m$ corresponding to x, that is

$$Dx - A^T\lambda + c = 0, \quad Ax \geq b, \quad \lambda \geq 0, \quad \lambda^T(Ax - b) = 0. \tag{2}$$

The KKT point set of (1) is denoted by $S(c, b)$. The solution set and the local solution set of (1) are denoted, respectively, by Sol(c, b) and loc(c, b). It is well-known (see Ref. 6, p. 115) that $S(c, b) \supset \text{loc}(c, b) \supset \text{Sol}(c, b)$. We are interested in studying the lower semicontinuity of the multifunction

$$S(\cdot, \cdot) : R^n \times R^m \to 2^{R^n}, \ (c', b') \mapsto S(c', b').$$

Note that lower semicontinuity properties of the multifunctions Sol$(\cdot, \cdot)$ and loc$(\cdot, \cdot)$, have been studied in Refs. 7 and 8.

Recall (Ref. 9, p. 451) that a multifunction $F : R^k \to 2^{R^n}$ is said to be lower semicontinuous (l.s.c.) at $\omega \in R^k$ if $F(\omega) \neq \emptyset$ and, for each open set $V \subset R^n$ satisfying $F(\omega) \cap V \neq \emptyset$, there exists $\delta > 0$ such that $F(\omega') \cap V \neq \emptyset$ for every $\omega' \in R^k$ with the property that $\|\omega' - \omega\| < \delta$. This definition differs slightly from the corresponding one given in Ref. 10, p. 39, where only the points from the effective domain of F are taken into account.

We obtain the necessary and sufficient conditions for the lower semicontinuity of the multifunction $S(\cdot, \cdot)$, our main results, in Section 2. Then, in Section 3, we consider several illustrative examples.

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