

# Multi-dimensional Limiting Process for Hyperbolic Conservation Laws

Sung-Hwan Yoon<sup>1</sup>, Kyu-Hong Kim<sup>1</sup> and Chongam Kim<sup>1</sup>

1) *School of Mechanical and Aerospace Engineering, Seoul National University, Seoul 151-744, KOREA*

Corresponding Author: Chongam Kim, e-mail@chongam.snu.ac.kr

## ABSTRACT

In this paper, we derive the three-dimensional limiting condition and present three-dimensional limiting process for hyperbolic conservation laws. The basic idea of multi-dimensional limiting condition is that the multi-dimensionally interpolated values at a vertex point should be within the maximum and minimum cell-average values of neighboring cells for the monotonic distribution. By applying the MLP (Multi-dimensional Limiting Process) to the three dimensional Euler and Navier-Stokes equations, we can achieve monotonic characteristics, which results in the enhancement of solution accuracy, convergence behavior.

## INTRODUCTION

Accurate monotonic schemes for hyperbolic conservation laws are developed based on one-dimensional flow physics through the analysis of TVD limiters [1], [2]. It shows the complete monotonic and accurate distribution in a one-dimensional discontinuity. However, if they are applied to a multi-dimensional problem, the interpolated property, without considering the effect of other flow directions, certainly leads to a non-monotonic distribution. In order to find out the monotonicity condition for multi-dimension, Kim et al. [3] extended the one-dimensional monotonic condition to two-dimensional problem and presented the two-dimensional limiting condition successfully. With the limiting condition, a multi-dimensional limiting process (MLP) is proposed which gives more accurate results for two-dimensional Euler and Navier-Stokes equations. It was the approach which prompted the work of the present paper. Basically, it extends the idea of MLP to three-dimensional problem. Thus, in this paper, we derive a three-dimensional limiting condition and present the multi-dimensional limiting process for three-dimensional problem.

## THREE-DIMENSIONAL LIMITING CONDITION

The basic criterion is that the multi-dimensionally interpolated values at a vertex point should be within the maximum and minimum cell-average values of neighboring cells for the monotonic distribution as follows.

$$\Phi_{\text{neighbor}}^{\min} < \Phi < \Phi_{\text{neighbor}}^{\max} \quad (1)$$

where  $\Phi$  is a vertex point value. In a three-dimensional problem, the vertex point values should be restricted as follows.

$$\min \begin{pmatrix} \Phi_{i,j,k}, & \Phi_{i+1,j,k}, & \Phi_{i,j-1,k}, & \Phi_{i+1,j-1,k}, \\ \Phi_{i,j,k+1}, & \Phi_{i+1,j,k+1}, & \Phi_{i,j-1,k+1}, & \Phi_{i+1,j-1,k+1} \end{pmatrix} < \Phi_1 < \max \begin{pmatrix} \Phi_{i,j,k}, & \Phi_{i+1,j,k}, & \Phi_{i,j-1,k}, & \Phi_{i+1,j-1,k}, \\ \Phi_{i,j,k+1}, & \Phi_{i+1,j,k+1}, & \Phi_{i,j-1,k+1}, & \Phi_{i+1,j-1,k+1} \end{pmatrix}$$

$$\min \begin{pmatrix} \Phi_{i,j,k}, & \Phi_{i+1,j,k}, & \Phi_{i,j+1,k}, & \Phi_{i+1,j+1,k} \\ \Phi_{i,j,k+1}, & \Phi_{i+1,j,k+1}, & \Phi_{i,j+1,k+1}, & \Phi_{i+1,j+1,k+1} \end{pmatrix} < \Phi_2 < \max \begin{pmatrix} \Phi_{i,j,k}, & \Phi_{i+1,j,k}, & \Phi_{i,j+1,k}, & \Phi_{i+1,j+1,k} \\ \Phi_{i,j,k+1}, & \Phi_{i+1,j,k+1}, & \Phi_{i,j+1,k+1}, & \Phi_{i+1,j+1,k+1} \end{pmatrix}$$

⋮

$$\min \begin{pmatrix} \Phi_{i,j,k}, & \Phi_{i-1,j,k}, & \Phi_{i,j-1,k}, & \Phi_{i-1,j-1,k} \\ \Phi_{i,j,k-1}, & \Phi_{i-1,j,k-1}, & \Phi_{i,j-1,k-1}, & \Phi_{i-1,j-1,k-1} \end{pmatrix} < \Phi_8 < \max \begin{pmatrix} \Phi_{i,j,k}, & \Phi_{i-1,j,k}, & \Phi_{i,j-1,k}, & \Phi_{i-1,j-1,k} \\ \Phi_{i,j,k-1}, & \Phi_{i-1,j,k-1}, & \Phi_{i,j-1,k-1}, & \Phi_{i-1,j-1,k-1} \end{pmatrix}$$

Now, let us consider the values at vertex points which are given as

$$\Phi = \Phi_{i,j,k} + \Delta\Phi_x^\pm + \Delta\Phi_y^\pm + \Delta\Phi_z^\pm \quad (2)$$

$\Delta\Phi_{x,y,z}^\pm$  are the variations from a center point to a cell-interface.  $\Delta\Phi_{x,y,z}^\pm$  is determined by the conventional TVD MUSCL limiter. From the information on multi-dimensional phenomena, the direction of the maximum gradient of property is defined and then we can derive a three-dimensional limiting condition by restricting  $\Delta\Phi_{x,y,z}^\pm$  to satisfy the condition of  $\Phi_{\text{neighbor}}^{\min} < \Phi < \Phi_{\text{neighbor}}^{\max}$ .

### MLP FOR THREE-DIMENSIONAL FLOWS

After the three-dimensional limiting condition is applied, the general form of MLP is written as follows

$$\Phi_{i+\frac{1}{2},L} = \Phi_i + 0.5 \max(0, \min(\alpha_L, \alpha_L r_L, \beta_L)) \Delta\Phi_{i-\frac{1}{2}} \quad (3A)$$

$$\Phi_{i+\frac{1}{2},R} = \Phi_{i+1} - 0.5 \max(0, \min(\alpha_R, \alpha_R r_R, \beta_R)) \Delta\Phi_{i+\frac{3}{2}} \quad (3B)$$

where  $\beta$  determines the type of limiting and  $\alpha$  is the multi-dimensional restriction coefficient as follows.

$$\alpha^+ = g \left( 2 \max(1, r_{i,j,k}^{-x}) \left[ \frac{\left( 1 + \frac{\tan \bar{\theta}_{i+1,j,k}}{r_{i+1,j,k}^{+x}} + \frac{1}{r_{i+1,j+1,k}^{+x} r_{i,j+1,k}^{+y}} \frac{\tan \bar{\phi}_{i+1,j+1,k}}{\cos \bar{\theta}_{i+1,j+1,k}} \frac{\tan \bar{\theta}_{i,j,k}}{\tan \bar{\theta}_{i,j+1,k}} \right)}{\left( 1 + \tan \theta + \frac{\tan \phi}{\cos \theta} \right)} \right] \right) \quad (4A)$$

$$\alpha^- = g \left( 2 \max(1, r_{i+1,j,k}^{+x}) \left[ \frac{\left( 1 + \frac{\tan \bar{\theta}_{i+1,j,k}}{r_{i,j,k}^{-x}} + \frac{1}{r_{i,j+1,k}^{-x} r_{i,j,k}^{-y}} \frac{\tan \bar{\phi}_{i+1,j+1,k}}{\cos \bar{\theta}_{i+1,j+1,k}} \frac{\tan \bar{\theta}_{i,j,k}}{\tan \bar{\theta}_{i,j+1,k}} \right)}{\left( 1 + \tan \theta + \frac{\tan \phi}{\cos \theta} \right)} \right] \right) \quad (4B)$$

It is noted that  $\alpha$  is a function of multi-dimensional flow parameters such as flow angle, cell-aspect ratio and local slopes.

### REFERENCES

1. P. K. Sweby : "High resolution schemes using flux limiters for hyperbolic conservation laws". SIAM J. Numer. Anal. Vol.21, No.5, Oct. 1984
2. A Harten, B.Engquist, S.Osher, and Chakravarthy," Uniformly high order accurate essentially non-oscillatory schemes," J. of Computational Physics, 71, 231-303 (1987)
3. K. H. Kim, and C. Kim : "Accurate, Efficient and Monotonic Numerical Methods for Multi-dimensional Compressible Flows" accepted to J. of Computational Physics, 2005