Space-Time Discretization of Hyperbolic Equations with Variable Collocation Points

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Abstract

In the field of scalar hyperbolic equations, we take into consideration the idea of building numerical schemes using two grids: the first one to represent the solution and the second one to collocate the equation. This approach is based on the experience already gained in the approximation of boundary-value elliptic-type problems (see for instance [1]), as well as in the field of functional ot integral-type equations (see [2]).

In order to show that the same approach can be used with success also for time-dependent problems, we study finite-difference approximations of first-order scalar hyperbolic equation in one space dimension. The representation grid is the usual uniform grid in the space-time plane. The discrete values of the solution are then assumed to be computed on a six-points stencil of such a representation grid. The approximating equations are deduced after collocation at a new point inside the stencil. It turns out that the possibility of varying the collocation point, gives a lot of freedom in the construction of the approximation method. First of all, this allows for the rediscovery of old methods and their critical analysis from a different point of view. Secondly, we have now the chance, by establishing a suitable relation between the

representation and the collocation grids, to introduce new methods.

Since the position of the collocation point (two degrees of freedom) characterizes the approximation scheme, we can come out with a wide family of finite-difference methods based on the six-points stencil. There will be actually three parameters after introducing another coefficient ν related to numerical viscosity. This family includes most of the classical linear schemes (implicit or explicit) for hyperbolic equations, such as the Crank-Nicolson or the Lax-Wendroff methods.

The results of a stability and consistency analysis are given and numerical examples show the performances of the different methods according to the choice of the parameters. We discuss experiments for some linear conservation laws. The problem of the determination of the parameters providing the best approximation is also addressed. In order to show that the idea can be adapted to more complicated problems, we also discuss some experiments for the non-linear Burgers equation. Generalizations to higher order methods (based on a larger stencil), or to different numerical techniques, can be, in principle, also taken into account. It is evident from the experiments that the qualitative behavior of the approximated solutions is quite sensitive to the choice of the parameters, so that, further theoretical improvements may concern with the detection of the "right way" to choose the parameters. This question has not unique answer, since it involves too many different aspects, such as the elimination of the oscillations, the preservation of the numerical accuracy far from the discontinuities, the minimization of the artificial viscosity, etc.

References

[1] D. Funaro, *Spectral Elements for Transport-Dominated Equations*, LNCSE, Vol. 1, Springer, 1997.

[2] D. Funaro, Superconsistent Discretizations of Integral Type Equations, Applied Numerical Mathematics, v.48 (2004), pp.1-11.