EULER–BERNOULLI BEAM WITH DYNAMIC FRICTIONLESS CONTACT

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ABSTRACT

In this work, we formulate the frictionless Euler–Bernoulli equation with dynamic contact condition along the length of a thin beam, and then set up a numerical formulation, employing the midpoint rule for the elastic part of the equation and the implicit Euler method for contact conditions. Convergence for our numerical formulation is investigated. The energy functional is defined, and our numerical scheme leads to energy dissipation. Using time discretization and the FEM with B-spline basis functions, we compute numerical solutions. In order to solve the linear complementarity problem that arises in the numerical method, we use a smoothed guarded Newton method. Those numerical schemes are implemented and some interesting numerical results are obtained. We also investigate numerically the question of whether the numerical solutions converge strongly to their limit, and if energy is conserved for the limit. Our numerical results give some evidence that this is so.

1 INTRODUCTION

The Euler–Bernoulli equation is an approximate equation for a thin beam such as a rod. Combining this with Signorini contact conditions will give our formulation of the Euler–Bernoulli beam with frictionless contact. The solution $u(x,t)$ of our formulation represents the vertical displacement of the rod from an initially horizontal position for time t and position x along the beam. The Euler–Bernoulli equation is represented by

$$\rho A \frac{\partial^2 u}{\partial t^2} = -EI \frac{\partial^4 u}{\partial x^4} + f(x,t) \quad \text{in } (0,L) \times (0,T],$$

where L is the length of the rod; A is the area of the cross section of the rod; ρ is the density of the rod; E is the Young modulus for the rod; and I is the second moment of inertia. Note that I is given by $I = \int_{\mathcal{A}} (y - \overline{y})^2 dx\, dy$, where $\mathcal{A}$ is the cross-section of the rod as a subset of the plane and $\overline{y}$ is the vertical center of area, which is $\overline{y} = \int_{\mathcal{A}} y\, dx\, dy / \int_{\mathcal{A}} dx\, dy$. The function $f(x,t)$ is the body force applied to the rod; and time t is between $t = 0$ and some fixed time $t = T$.

Suppose that the end of the rod at $x = 0$ is clamped horizontally. Then the boundary condition of $x = 0$ has the homogeneous *essential* boundary conditions, i.e., $u(0,t) = u_x(0,t) = 0$. If the end of rod is free, then we have the *natural* boundary conditions $u_{xx}(L,t) = u_{xxx}(L,t) = 0$.

If we impose frictionless Signorini contact conditions along the length of the rod, we represent the equation of motion as

$$\rho A \frac{\partial^2 u}{\partial t^2} = -EI \frac{\partial^4 u}{\partial x^4} + f(x,t) + N(x,t), \tag{1}$$

where from contact criterion, the magnitude of the vertical contact forces (pressures), $N(x,t)$ satisfy the linear complementary problem (LCP)

$$0 \leq N(x,t) \quad \perp \quad u(x,t) + g(x) \geq 0. \tag{2}$$

Note that $g(x)$, called *gap function*, displays a measure of the initial normalized "gap" between the rod and the rigid foundation, where the position of rod is on the same as its clamped point of the rod horizontally. Thus we are led to consider solving the following PDE:

$$\rho A u_{tt} = -EI u_{xxxx} + f(x) + N(x,t) \quad \text{in } (0,L) \times (0,T], \tag{3}$$

$$0 \leq N(x,t) \perp u(x,t) + g(x) \geq 0 \quad \text{in } (0,L) \times (0,T] \tag{4}$$

$$u(0,t) = u_x(0,t) = 0, u_{xx}(L,t) = u_{xxx}(L,t) = 0 \quad \text{on } (0,T], \tag{5}$$

$$u(x,0) = u^0(x), u_t(x,0) = v^0(x) \quad \text{in } (0,L), \tag{6}$$

We will assume that $f \in L^2(0,L)$, $u^0 \in H^2_{cf}(0,L)$, $v^0 \in L^2(0,L)$, $g \in C[0,L]$, and that $g(0) > 0$. Note that $H^2_{cf}(0,L)$ is the subset of $H^2(0,L)$ which satisfies the clamped end condition at $x = 0$ ("c" denotes "clamped", while "f" denotes "free"). We focus on a numerical approach to the PDE, since the existence of solutions to the PDE has been already shown in [1].

2 NUMERICAL FORMULATION OF THE DISCRETE-TIME PROBLEM

In order to set up a numerical formulation, we will employ the two numerical schemes on the time space: 1.Elasticity (u_{xxxx}) - Midpoint rule is used 2.Contact condition - Implicit Euler is used. After partitioning time, we denote by $u^l(x)$ numerical solution of displacement $u(x,t_l)$ and by $v^l(x)$ numerical solution of velocity $v(x,t_l)$ and $N^l(x)$ numerical solution of magnitude of contact force, $N(x,t_l)$, respectively at each discretized time $t_l = lh$. Then the time step size is $h = t_{l+1} - t_l$, for $l \geq 0$. From (3), we take $\rho A = EI = 1$ by proper scaling.

Using our numerical scheme we establish numerical formulation:

$$\frac{v^{l+1}(x) - v^l(x)}{h} = -\left(\frac{u^{l+1}_{xxxx}(x) + u^l_{xxxx}(x)}{2}\right) + f(x) + N^l(x) \tag{7}$$

$$\frac{u^{l+1}(x) - u^l(x)}{h} = \frac{v^{l+1}(x) + v^l(x)}{2} \tag{8}$$

$$0 \leq N^l(x) \quad \perp \quad u^{l+1}(x) + g(x) \geq 0. \tag{9}$$

Employing the numerical formulations (7–9), we investigate convergence and implement numerical schemes, and present numerical results (simulation).

REFERENCES

(1) Paper in a journal

1. Ahn, Jeongho, and Stewart, David, "An Euler–Bernoulli Beam in Impact: Existence", submitted to *Applied Mathematics and Optimization*