PIECEWISE BILINEAR PRECONDITIONING ON HIGH-ORDER FINITE ELEMENT METHOD *

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Abstract. The high-order finite element method for discretization to solve a second-order uniformly elliptic partial differential equation leads to a linear equation $\hat{L}_{N^2}U = F$ which needs efficient iterative methods such as Schwarz-based methods, preconditioning methods related to multilevel methods, multigrid methods and etc. This is because such linear systems have large condition numbers usually dependent on degrees of high-order elements and mesh sizes. In particular, an algebraic multigrid (AMG) method is useful in the case of irregular grids like Gauss type points, but it was reported that a direct application of AMG to $\hat{L}_{N^2}U = F$ is not so efficient. Hence one may need bilinear finite elements preconditioning iterative methods or AMG preconditioning based on bilinear finite elements in a conjugate gradient method, which has the advantage of a straightforward matrix-free implementation, to solve a linear system arisen from high-order finite element discretizations based on Legendre-Gauss-Lobatto quadrature points for a model problem such that

(0.1) $$L_C u := -\nabla \cdot \mathbf{C}\nabla u + cu = f \quad \text{in} \quad \Omega = (-1,1)\times(-1,1)$$

with boundary conditions

(0.2) $$u = 0 \quad \text{on} \quad \Gamma_D(L_C), \quad \mathbf{n}\cdot\mathbf{C}\nabla u = \alpha_c u \quad \text{on} \quad \Gamma_N(L_C)$$

where $\Gamma_{L_C} = \Gamma_D(L_C) \cup \Gamma_N(L_C)$ and $\Gamma_D(L_C)$ is a nonempty set. In this paper, we consider a piecewise bilinear finite element preconditioner corresponding to an operator $L_{\mathbf{B}}$ as in (0.1) and (0.2) which leads to a linear system $\hat{B}_{h^2}$ to reduce the condition numbers of $\hat{L}_{N^2}$ induced by high-order elements. As addressed in [**?**], we suppose that two Dirichlet boundary conditions $\Gamma_D(L_C)$ and $\Gamma_D(L_B)$ are same but we allow $\Gamma_N(L_B) \neq \Gamma_N(L_B)$. We assume that all matrices appeared in this paper are uniformly positive definite matrices: *For example,* the matrix $\mathbf{C}$ satisfies

(0.3) $$0 < \lambda \boldsymbol{\xi}^t\boldsymbol{\xi} \leq \boldsymbol{\xi}^t\mathbf{C}(x,y)\boldsymbol{\xi} \leq \Lambda\boldsymbol{\xi}^t\boldsymbol{\xi} < \infty \quad \text{for all} \quad \boldsymbol{\xi}\in\Re^2 \quad \text{and almost all} \quad (x,y)\in\bar{\Omega}.$$

The numerical results in [**?**] show that the condition number $(\hat{B}_{h^2})^{-1}\hat{L}_{N^2}$ is bounded. For a single spectral element, this kind of preconditioning was analyzed in Legendre spectral collocation methods in many literatures. The main object in this article is to prove that the condition numbers $(\hat{B}_{h^2})^{-1}\hat{L}_{N^2}$ are not dependent on the degrees of high-order elements and the mesh sizes. These results are motivated by the desire to find multigrid algorithms for solving problems like (0.1) with high-order elements discretizations, which guarantee convergence of the strategy of preconditioning the high-order matrix with a bilinear or trilinear matrix based on Legendre-Gauss-Lobatto quadrature nodes well suited to solution by multigird methods. The goal of this paper can be realized by extending the results of Parter and Rothman to high-order elements and by applying H^1, L_2 estimates in Widlund of a local interpolation operator $\mathcal{I}_N$ to a global interpolation operator $\mathcal{I}_N^h$. Further we note that such an H^1 semi-norm estimate of the local interpolation operator defined on a space of piecewise linear can be extended to the space of H^1 by modifying the relevant results in Bernardi and Maday. We also note that the discussions here can be extended to singular value results for more general elliptic operators which are not positive definite. For this, one had better refer to Parter's work.

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