Butterworth Filters, Scaling functions and Frame Wavelets

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ABSTRACT

We give a regularity analysis of the Butterworth scaling function and provide various constructions of frame wavelets via the unitary extension principle. In general, there is no precise regularity estimate for Butterworth scaling function. Here, an optimal estimate of the decay of the Fourier transform of the Butterworth scaling function is given. The regularity of the Butterworth scaling function can then be deduced and hence, the regularity of the corresponding frame wavelets. We also show that the Butterworth scaling function tend to the Shannon scaling function as the Butterworth filter goes to infinity.

BUTTERWORTH FRAME WAVELETS VIA UEP

Ron and Shen introduced the concept of the Unitary Extension Principle in order to find tight frame wavelets as follows:

Theorem[UEP] Given $H_0 \in L^\infty(\mathbb{T})$ with $H_0(0) = 1$ such that $\widehat{\psi}_0(w) = \prod_{j=1}^{\infty} H_0(w/2^j)$ converges to $L^2(\mathbb{R})$ function, find $H_1, H_2, \cdots, H_n \in L^\infty(\mathbb{T})$ such that

$$\sum_{l=0}^{n} |H_l(w)|^2 = 1,$$
$$\sum_{l=0}^{n} H_l(w)\overline{H_l(w+\pi)} = 0,$$

for *a.e.* $w \in \mathbb{T}$. Define

$$\widehat{\psi_l}(w) = H_l(w/2)\widehat{\psi_0}(w/2), \;\; l = 1, \cdots, n.$$

Then, the family $\{2^{j/2}\psi_l(2^j \cdot - k)\}_{j,k\in\mathbb{Z},l=1,\cdots,n}$ constitutes a tight frame for $L^2(\mathbb{R})$ with frame bound equal to 1.

We are concerned only with interesting examples of multiwavelet frames with rational filters. We apply UEP to the spectral decompositions of the Butterworth-type filters of the form

$$\left(\frac{\cos^{2n}(w/2)}{\cos^{2n}(w/2) + \sin^{2n}(w/2)}\right)^K \tag{1}$$

to construct the frame wavelets of K generators.

REGULARITY AND LIMIT

Let H_0 be the spectral decompositions of the Butterworth-type filter defined as in (1). Define ψ_0 as above. Then we obtain the following 'optimal' estimate of the decay of the Fourier transform of the Butterworth scaling function:

$$|\hat{\psi}_0(w)| \leq C(1+|w|)^{-(2-\log_2(4/3))Kn/2}.$$

Hence the smoothness of the Butterworth scaling function can then be deduced:

$$\psi_0 \in C^{(2-\log_2(4/3))(Kn/2)-1-\epsilon}.$$

We also show that the Butterworth scaling function tends, in $L^q(\mathbb{R})(2 \leq q \leq \infty)$, in particular uniformly, to the Shannon scaling function φ_{SH} as the order n of H_0 approaches to the infinity, where

$$\hat{\varphi}_{SH}(w) := \chi_{[-\pi,\pi]}(w).$$

REFERENCES

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