THE FINITE ELEMENT METHOD DEALING WITH CORNER SINGULARITIES : DIV-CURL SYSTEM

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ABSTRACT

In [1–3] we give a new finite element method to control the domain singularity of the solution of Poisson equation. Years ago we reported that the singular decomposition of the solution of Div-curl system is computed. Now we find the dual singular functions and applied our method to control the domain singularity of the Div-curl system and get the expected results.

MODEL PROBLEMS

We consider two Model problems;

(1) The Poisson equation with the Dirichlet boundary condition in a non-convex polygon $\Omega \in R^2$:

$$\begin{cases} -\Delta u = f, \text{ in } \Omega, \\ \quad u = 0, \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where Δ stands for the Laplacian operator, f is a given function in $L^2(\Omega)$, and Ω is an open, bounded polygonal domain in R^2.

(2) The Div-curl system in a non-convex polygon $\Omega \subset R^2$:

$$\begin{cases} (I - \nabla\nabla\cdot + \nabla^\perp\nabla\times)\mathbf{u} = \mathbf{f} \text{ in } \Omega, \\ \mathbf{n}\cdot\mathbf{u} = 0 \text{ on } \Gamma_N, \\ \nabla\times\mathbf{u} = 0 \text{ on } \Gamma_N, \\ \boldsymbol{\tau}\cdot\mathbf{u} = 0 \text{ on } \Gamma_D, \\ \nabla\cdot\mathbf{u} = 0 \text{ on } \Gamma_D. \end{cases} \tag{2}$$

Here $\mathbf{f} \in (L^2(\Omega))^2$ and the curl operator in $\mathbf{R}^2$ and its formal adjoint are

$$\nabla\times = (-\partial_2, \partial_1), \quad \nabla^\perp = \begin{pmatrix} \partial_2 \\ -\partial_1 \end{pmatrix}.$$

For simplicity assume Ω have only one re-entrant angle.

SPACES AND STANDARD WEAK FORMULATION

Consider

$$H(\text{div}\,;\Omega) = \{\mathbf{v} \in L^2(\Omega)^2 :\ \nabla \cdot \mathbf{v} \in L^2(\Omega)\},$$
$$H_N(\text{div}\,;\Omega) = \{\mathbf{v} \in H(\text{div}\,;\Omega) : \mathbf{n} \cdot \mathbf{u} = 0 \text{ on } \Gamma_N\},$$
$$H(\text{curl};\Omega) = \{\mathbf{v} \in L^2(\Omega)^2 :\ \nabla\times\mathbf{v} \in L^2(\Omega)\},$$
$$H_D(\text{curl};\Omega) = \{\mathbf{v} \in H(\text{curl};\Omega) : \tau \cdot \mathbf{u} = 0 \text{ on } \Gamma_D\},$$

and

$$\mathcal{V} = H_N(\text{div}\,;\Omega) \cap H_D(\text{curl};\Omega),$$

$$\|\mathbf{v}\|_{\mathcal{V}} = \left(\|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2 + \|\nabla\times\mathbf{v}\|^2\right)^{\frac{1}{2}}.$$

The weak formulation is (WF); Find $\mathbf{u} \in \mathcal{V}$ such that

$$a(\mathbf{u},\mathbf{v}) := (\mathbf{u},\mathbf{v}) + (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) + (\nabla\times\mathbf{u}, \nabla\times\mathbf{v}) = (\mathbf{f},\mathbf{v}), \qquad \forall \mathbf{v} \in \mathcal{V}. \tag{3}$$

REGULARITY AND CONVERGENCE

Note if the inner angle is ω, $\frac{\pi}{2} < \omega$, and the boundary condition changes at that corner, then the solution belongs to $H^{\frac{\pi}{2\omega}-\epsilon}$. For example, if $\omega = \frac{3\pi}{2}$, then the solution belongs only to $H^{\frac{1}{3}-\epsilon}$. If we use the standard finite element method to the above (WF), we will get only $h^{1+\frac{1}{3}}$ convergence in L^2-norm and $h^{\frac{1}{3}}$ convergence in H^1-norm.

MAIN RESULTS

In [1–3] we give a new finite element method to control the singularity of the solution of Model Problem (1). Recently we report that the singular decomposition of the solution of Model Problem (2) is computed. We applied the method to the problem Model Problem (2) and get the expected results.

REFERENCES

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