The Implicit Immersed Finite Element Method for Fluid-Solid Interaction Problems

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ABSTRACT

In this paper, we propose an immersed solid system method to treat efficiently the fluid-structure interaction problems. Augmenting a fluid in the moving solid domain, we introduce a volumetric force to obtain the correct dynamics for both the fluid and the structure. We further define an Euler-Lagrange mapping to describe the motion of the immersed solid. A weak formulation is then constructed and shown to be equivalent to both the fluid-structure interaction and the immersed solid system under certain regularity assumptions. The weak formulation may be computed numerically by an implicit algorithm with the finite element method, and the SUPG (streamline upwind/Petrov Galerkin) method. Compared with the successful immersed boundary method(IBM) by Peskin [1–3], the immersed solid system method applies to more general geometries with non-uniform grids and avoids the inaccuracy in approximating the Dirac delta function.

FLUID-SOLID INTERACTION PROBLEMS

Consider the fluid-structure system in an open bounded domain $\Omega \subset \mathbb{R}^d\,(d = 2,\,3)$. At time t, the solid occupies an open subdomain Ω_t^s. We further assume the boundaries $\Gamma \equiv \partial\Omega$ and $\Gamma_t^s \equiv \partial\Omega_t^s$ to be smooth, and do not intersect with each other.

For the fluid in $\Omega \setminus \overline{\Omega_t^s}$, we assume the density ρ^f to be constant. The primary variables are its velocity $\mathbf{v}^f$ and pressure p^f. The governing equations are usually expressed in terms of material derivative. The solid is described by the Lagrangian coordinate. The primary variables are $\mathbf{x}(\mathbf{X}, t) \in \Omega_t^s$, denoting the position at time t for a particle that initially lies at $\mathbf{X} \in \Omega_0^s$; the solid mean stress p^s; and the velocity $\mathbf{v}^s$.

Table FSI: Governing equations for fluid-structure interaction system
Solution: $(\,\mathbf{v}^f(\mathbf{x}, t),\ p^f(\mathbf{x}, t),\ \mathbf{v}^s(\mathbf{x}, t),\ p^s(\mathbf{x}, t),\ \mathbf{x}(\mathbf{X}, t)\,)$

Fluid (in $\Omega \setminus \overline{\Omega_t^s}$)

continuity
$$\nabla \cdot \mathbf{v}^f + \frac{1}{\kappa^f}\dot{p}^f = 0 \tag{1}$$

momentum
$$\rho^f\,\dot{\mathbf{v}}^f = \nabla \cdot \boldsymbol{\sigma}^f + \rho^f\,\mathbf{g} \tag{2}$$

stress
$$\boldsymbol{\sigma}^f = -p^f\,\mathbf{I} + \lambda(\nabla \cdot \mathbf{v}^f)\,\mathbf{I} + \mu\,\left[\nabla\mathbf{v}^f + (\nabla\mathbf{v}^f)^T\right] \tag{3}$$

Solid (in Ω_t^s)

mapping
$$\mathbf{v}^s(\mathbf{x}(\mathbf{X}, t), t) = \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t} \tag{4}$$

mean stress $$\left|\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right| - 1 + \frac{1}{k^s} p^s = 0 \tag{5}$$

dynamics $$\rho^s \, \dot{\mathbf{v}}^s = \nabla \cdot \boldsymbol{\sigma}^s + \rho^s \, \mathbf{g} \tag{6}$$

stress $$\boldsymbol{\sigma}^s = -p^s \, \mathbf{I} + \left|\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right|^{-1} \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \left(2(c_1 + c_2 \ \mathrm{Tr}(\mathbf{C}))\, \mathbf{I} - 2c_2 \, \mathbf{C}\right) \left[\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right]^T \tag{7}$$

Interface (on Γ_t^s)

no-slip condition $$\mathbf{v}^f(\mathbf{x}(\mathbf{X}, t), t) = \mathbf{v}^s(\mathbf{x}(\mathbf{X}, t), t) \tag{8}$$

normal stress $$\boldsymbol{\sigma}^f \cdot \mathbf{n} = \boldsymbol{\sigma}^s \cdot \mathbf{n} \tag{9}$$

Initial and boundary conditions

fluid $$(\mathbf{v}^f(\mathbf{x}, 0), p^f(\mathbf{x}, 0)) = (\mathbf{v}_0^f(\mathbf{x}), p_0^f(\mathbf{x})), \quad \mathbf{x} \in \Omega \setminus \Omega_t^s \tag{10}$$

$$\mathbf{v}^f(\mathbf{x}, t) = \mathbf{v}_B^f(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega \tag{11}$$

solid $$\mathbf{v}^s(\mathbf{X}, 0) = \mathbf{v}_0^s(\mathbf{X}), \qquad \mathbf{x}(\mathbf{X}, 0) = \mathbf{X}, \quad \mathbf{X} \in \Omega_0^s, \tag{12}$$

THE IMMERSED FINITE ELEMENT METHOD

The immersed finite element method is developed for the purpose of solving the above FSI problem.[4–6] In this paper, we will describe an immersed solid system (ISS) equivalent to the fluid-structure interaction problem. We discover that the fluid-solid interaction force could be interpreted as a volumetric force applied on the entire region of the structure in the augmented fluid. Using the interaction force derived in a weak form, we derive a weak formulation of the immersed solid system. We further define an Euler-Lagrange mapping to describe the motion of the solid structure. In particular, this allows us to circumvent the moving domain for integration. Under appropriate regularity condition, we show that the weak formulation is equivalent to the immersed solid system.

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