The Boltzmann equation near a local Maxwellian

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ABSTRACT

We present two robust nonlinear functionals measuring future possible collisions and L^1-distance between two mild solutions to the Boltzmann equation near a close-to-a local Maxwellian regime. Through the explicit local cancellation of the collision mechanism due to the special structure of a local Maxwellian, we show that the nonlinear functionals in [4] satisfy stability estimates. These functionals can be employed to the study of L^1-scattering and uniform L^1-stability estimate of mild solutions for the moderately soft interaction potential.

DESCRIPTION OF MAIN RESULTS

The Boltzmann equation describes the statistical evolution of a velocity distribution function $f = f(x, \xi, t)$ of moderately rarefied gases. In the absence of external forces, the velocity distribution function f satisfies

$$\begin{aligned} &\partial_t f + \xi \cdot \nabla_x f = Q(f, f), \quad (x, \xi, t) \in R^3 \times R^3 \times R_+, \\ &f(x, \xi, 0) = f_0(x, \xi), \end{aligned} \tag{1}$$

where $Q(f, f)$ is a quadratic collision operator which only acts on the velocity variable ξ, and reads as

$$Q(f, f)(\xi) \equiv \frac{1}{\kappa} \int_{R^3 \times S^2_+} B(\xi - \xi_*, \omega)(f' f'_* - f f_*) d\omega d\xi_*. \tag{2}$$

Here κ is a Knudsen number which is a ratio between the mean free path of molecules and the characteristic length of the flow, $S^2_+ \equiv \{\omega \in S^2 : (\xi - \xi_*) \cdot \omega \geq 0\}$, and we used abbreviated notations:

$$f' \equiv f(x, \xi', t), \quad f'_* \equiv f(x, \xi'_*, t), \quad f \equiv f(x, \xi, t) \quad \text{and} \quad f_* \equiv f(x, \xi_*, t).$$

On the other hand, scattered velocities (ξ', ξ'_*) are given by the incident velocities (ξ, ξ_*) and $\omega \in S^2_+$:

$$\xi' = \xi - [(\xi - \xi_*) \cdot \omega]\omega \quad \text{and} \quad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \omega]\omega. \tag{3}$$

The purpose of this talk is to show that the nonlinear functional approach in [4] can be used in the regime of a close to a local Maxwellian $\mathcal{M}(x, \xi)$:

$$\mathcal{M}(x, \xi) \equiv c e^{-\alpha|x|^2 - \beta|\xi|^2}, \qquad \alpha, \beta > 0. \tag{4}$$

Without loss of generality, we may assume $c = 1$. This local Maxwellian has the crucial key properties:

- Its travelling profile $\mathcal{M}(x - t\xi, \xi)$ is a solution to (1).

- $\mathcal{M}$ satisfies

$$\mathcal{M}(x+t(\xi-\xi_*'),\xi_*')\mathcal{M}(x+t(\xi-\xi'),\xi') = \mathcal{M}(x+t\xi,\xi)\mathcal{M}(x+t(\xi-\xi_*),\xi_*).$$

So far, most available estimates in [2] on the Boltzmann equation near vacuum regime use the domination by that of the transport equation with only gain operator:

$$\partial_t + \xi\cdot\nabla_x f = Q_+(f,f), \tag{5}$$

where $Q_+(f,f)$ is the gain operator defined as

$$Q_+(f,f) \equiv \frac{1}{\kappa}\int_{R^3\times S^2_+} B(\xi-\xi_*,\omega) f' f_*' d\omega d\xi_*.$$

Hence it is obvious that the cancellation effects between the gain and loss operators are ignored, therefore it cannot be implemented in the regime of far from vacuum [1].

The novelty of this work is that we use the special structure of local steady Maxwellian $\mathcal{M}$ to see the global cancellation of the collision operator near $\mathcal{M}$. Below we briefly outline of the main results given in this paper. Since the existence of classical solutions in a close-to-$\mathcal{M}$ is not known, we cannot use the equation (1) directly, but this difficulty can be bypassed using the mollified Boltzmann equations:

$$\partial_t f_\varepsilon + \xi\cdot\nabla_x f_\varepsilon = Q(f_\varepsilon,f_\varepsilon) + P(f,f_\varepsilon),$$

where f_ε is the mollification of mild solution f and

$$P(f,f_\varepsilon) \equiv Q_\varepsilon(f,f) - Q(f_\varepsilon,f_\varepsilon).$$

We next list the main assumptions $(\mathcal{A}1)-(\mathcal{A}2)$ employed in this work:

- $(\mathcal{A}1)$. The collision kernel satisfies an inverse power potential and an angular cut-off assumption:

$$B(\xi-\xi_*,\omega) = |\xi-\xi_*|^\gamma b_\gamma(\theta), \quad -2<\gamma\le 1 \quad \text{and} \quad \frac{b_\gamma(\theta)}{\cos\theta} \le B_* < \infty,$$

  where θ is the angle between $\xi-\xi_*$ and ω.
- $(\mathcal{A}2)$. initial datum f_0 satisfies

$$f_0 \in C(R^6), \qquad (1-\delta_0)\mathcal{M}(x,\xi) \le f_0(x,\xi) \le (1+\delta_0)\mathcal{M}(x,\xi),$$

  whre $\delta_0 \ll \kappa$ is a positive constant.

The main results of this work are the stability estimates of two nonlinear functionals: First we present the collision potential $\mathcal{D}(f_\varepsilon(t))$ introduced in [4] evaluated along the mollification f_ε:

$$\mathcal{D}(f_\varepsilon(t)) \equiv \int_{R^3\times R^3} f_\varepsilon^\sharp(x,\xi,t) \times \Big[\int_{R^3\times R_+} |\xi-\xi_*|^{\gamma-1} f_\varepsilon^\sharp(x+t(\xi-\xi_*)+\tau n(\xi,\xi_*),\xi_*,t)d\tau d\xi_*\Big] d\xi dx,$$

where $n(\xi, \xi_*)$ denotes the unit vector in the direction of $\xi - \xi_*$. By detailed analysis, we show that $\mathcal{D}(f_\varepsilon)$ satisfies

$$\frac{d}{dt}\mathcal{D}(f_\varepsilon(t)) \leq -C_0\Lambda(\mathcal{M}_\varepsilon) + o(\varepsilon) \qquad \text{as } \varepsilon \to 0,$$

where $\Lambda(\mathcal{M}_\varepsilon)$ is the collision production rate. In the $\varepsilon \to 0$, we recover a Lyapunov estimate:

$$\frac{d}{dt}\mathcal{D}(f(t)) \leq -C_0\Lambda(\mathcal{M}).$$

Secondly, we present a nonlinear functional $\mathcal{H}^\varepsilon(t)$:

$$\begin{aligned}\mathcal{H}^\varepsilon(t) = &\int_{R^6} |f_\varepsilon - \bar{f}_\varepsilon|^\sharp(x,\xi,t) \\ &\times \left[1 + \int_{R^3\times R_+} |\xi - \xi_*|^{\gamma-1}(f_\varepsilon^\sharp + \bar{f}_\varepsilon^\sharp)(x + t(\xi-\xi_*) + \tau n(\xi,\xi_*), \xi_*, t)d\tau d\xi_*\right] d\xi dx.\end{aligned}$$

This functional is equivalent to the L^1-distance between two mild solutions in the sense that

$$||f(t) - \bar{f}(t)||_{L^1} \leq \mathcal{H}(t) \leq C_1||f(t) - \bar{f}(t)||_{L^1},$$

where C_1 is a positive constant independent of t. Through the detailed pointwise estimates of local Maxwellian $\mathcal{M}$, we show that the functional satisfies stability estimate

$$\frac{d}{dt}\mathcal{H}^\varepsilon(t) + \Lambda_d^\varepsilon(t) \leq \mathcal{O}((t+1)^{-(\gamma+3)})\mathcal{H}^\varepsilon(t) + o(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

This implies a stability estimate via Gronwall's estimate

$$\mathcal{H}(t) + \int_0^t \Lambda_d(s)ds \leq C_1\mathcal{H}(0) \quad t \geq 0,$$

which yields L^1-stability of mild solutions.

$$||f(t) - \bar{f}(t)||_{L^1} \leq \mathcal{H}(t) \leq C_1\mathcal{H}(0) - \int_0^t \Lambda_d(s)ds \leq C_1C_0||f_0 - \bar{f}_0||_{L^1} - \int_0^t \Lambda_d(s)ds.$$

For more detailed reference, we refer to authors' recent preprint [3].

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