Explicit solutions of a convection-reaction equation and defects of difference schemes

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Abstract

We introduce two kinds of explicit solutions to the convection-reaction equation,

$$u_t + (|u|^q/q)_x = u, \qquad u, x \in \mathbf{R},\ t \in \mathbf{R}^+,\ q > 1,$$

and employ them to test properties of various computational schemes. From this test we observe that computed solutions using Lax-Friedrichs, MacCormack and Lax-Wendroff schemes break down in a finite time. On the other hand some other schemes including WENO, NT and Godunov show more stable behavior and the tests provide their detailed behaviors. It is discussed that if a numerical scheme is applied to this problem together with the splitting method, certain defects of the scheme can be magnified exponentially and observed easily. Sometimes such a behavior destroys the numerical solution completely and hence one need to pay extra caution to deal with reaction dominant systems.

1 Introduction

The main goal of this paper is to develop a method to survey the property of numerical schemes. Our method starts with a Cauchy problem for a convection-reaction equation

$$u_t + uu_x = u, \quad u(x,0) = u_0(x) \in L^1(\mathbf{R}). \tag{1.1}$$

This equation models the roll wave [1] and several analytic properties have been studied in [2,3].

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We introduce a more general case

$$u_t + f(u)_x = u, \quad u(x,0) = u_0(x) \in L^1(\mathbf{R}), \qquad x, u \in \mathbf{R}, \quad t > 0, \quad (1.2)$$

where the flux is given by the convex power law

$$f(u) = \frac{1}{q}|u|^q, \qquad q > 1. \quad (1.3)$$

In Section 2, we derive two kinds of explicit solutions of the problem which enable us to compare computed solutions with an exact one.

In the rest of the paper several numerical schemes are tested comparing with the exact solutions of the convection-reaction equation (1.2).

References

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