ON THE 2D G-NAVIER-STOKES EQUATIONS

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1. Introduction

The g-Navier-Stokes equations has the following form,

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p \;=\; f \;\text{ in }\; \Omega\times(0,\infty), \tag{1.1}$$

$$\frac{1}{g}\nabla\cdot(gu) = \;0 \;\text{ in }\; \Omega\times(0,\infty), \tag{1.2}$$

$$u(\cdot,0) = u_0(\cdot) \;\text{ in }\; \Omega, \tag{1.3}$$

where $g = g(x_1, x_2)$ is a suitable real-valued smooth function. One note that the g-Navier-Stokes equations become the Navier-Stokes equations for $g = 1$. For many years, the Navier-Stokes equations were investigated by many authors and the existence of the attractors for 2D Navier-Stokes equations was first proved by Ladyzhenskaya[3] and independently by Foias and Temam[2]. For the analysis on the Navier–Stokes equations, one can refer to [1] and [10], specially [11] for the periodic boundary conditions. In this talk, I will present the dynamics of the g-Navier-Stokes equations on the bounded domains([4], [5], [6]) and eventhough we can get some results on the unbounded domain([7], [8]). One can also refer [9] for the dimensions of the global attractor of the g-Navier-Stokes equations. Then, finally I will talk about the g-Navier-Stokes equations when the function g depends on time t.

2. Main Results

Here, we consider the periodic boundary conditions on the domain $\Omega = (0,1)\times(0,1)$ and for the function g, throughout this paper, we assume that $g(\mathbf{x}) \in C^{\infty}_{per}(\Omega)$ and $0 < m \le g(x,y) \le M$, for all $(x,y)\in\Omega$.

Now, we define the Hilbert space $L^2_{per}(\Omega, g) = L^2_{per}(\Omega, R^2, g)$, which is the space $L^2_{per}(\Omega)$ with the scalar product and the norm given by

$$< \mathbf{u}, \mathbf{v} >_g = \int_\Omega (\mathbf{u} \cdot \mathbf{v})\ g\ d\mathbf{x} \quad \text{and} \quad \| \mathbf{u} \|^{\mathbf{2}}_{\mathbf{g}} = < \mathbf{u}, \mathbf{u} >_{\mathbf{g}},$$

where $\mathbf{x} = (x_1, x_2)$. Similarly, we define the space $H^1(\Omega, g)$ which is the space $H^1(\Omega)$ with the norm by

$$\| \mathbf{u} \|_{H^1(\Omega,g)} = [\langle \mathbf{u}, \mathbf{u} \rangle_g + \sum_{i=1}^{2} \langle D_i \mathbf{u}, D_i \mathbf{u} \rangle_g]^{\frac{1}{2}}.$$

For our problem, we consider the periodic boundary conditions and are interested in the dynamics on the following spaces;

$$\begin{aligned} H_g &= CL_{L^2(\Omega,g)}\{\mathbf{u} \in C^\infty_{per}(\Omega)\ :\ \nabla \cdot g\mathbf{u} = 0,\ \int_\Omega \mathbf{u}\ d\mathbf{x} = \mathbf{0}\} \\ V_g &= \{\mathbf{u} \in H^1_{per}(\Omega, g)\ :\ \nabla \cdot g\mathbf{u} = 0,\ \int_\Omega \mathbf{u}\ d\mathbf{x} = \mathbf{0}\} \\ Q &= CL_{L^2(\Omega,g)}\{\nabla \phi : \phi \in C^1_{per}(\bar{\Omega}, R)\}. \end{aligned}$$

Then, we can define the orthogonal projection $P_g : L^2_{per}(\Omega, g) \mapsto H_g$, which is similar to the Lerary projection.

In this paper, we define $\Delta_g \mathbf{u}$ as

$$-\Delta_g \mathbf{u} = -\frac{1}{g}(\nabla \cdot g \nabla)\mathbf{u} = -\Delta \mathbf{u} - \frac{1}{g}(\nabla g \cdot \nabla)\mathbf{u},$$

which is a perturbation of $-\Delta \mathbf{u}$. Then, for $\nu = 1$, (1.1) can be written as

$$\frac{\partial \mathbf{u}}{\partial t} - \Delta_g \mathbf{u} + \frac{1}{g}(\nabla g \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \qquad \text{in} \qquad \Omega. \tag{2.1}$$

So, by taking the orthogonal projection P_g into (2.1), one obtains

$$\frac{d\mathbf{u}}{dt} + A_g \mathbf{u} + B_g(\mathbf{u}, \mathbf{u}) = \mathbf{q} \qquad \text{on} \qquad H_g, \tag{2.2}$$

where $A_g \mathbf{u} = P_g(-\Delta_g \mathbf{u})$, $B_g(\mathbf{u}, \mathbf{u}) = P_g(\mathbf{u} \cdot \nabla)\mathbf{u}$, $\mathbf{q} = P_g[\mathbf{f} - \frac{1}{g}(\nabla g \cdot \nabla)\mathbf{u}]$.

Theorem 2.1. *Let* $\mathbf{f} \in L^2(0, \infty; L^2(\Omega, g))$ *is given. Then for every* $\mathbf{u}_0 \in H_g$ *there is precisely one weak solution (of class LH)* $\mathbf{u} = \mathbf{u}(t)$ *on* $[0, \infty)$ *of (2.2), satisfying* $\mathbf{u}(0) = \mathbf{u}_0$*. Also, let* $\mathbf{u} = \mathbf{u}(t)$ *be any weak solution of (2.2) on* $[0, \infty)$ *with initial condition* $\mathbf{u}(0) = \mathbf{u}_0 \in H_g$*. Then for each* $t_0 > 0$*,* $\mathbf{v}(t) = \mathbf{u}(t + t_0)$ *is a strong solution of (2.2) on* $[0, \infty)$ *with initial condition* $\mathbf{v}(0) = \mathbf{u}(t_0)$ *and* $D_t \mathbf{u} \in L^2_{loc}(0, \infty; H_g)$*.*

We define $\tilde{\sigma}_w(g,\mathbf{v},t)$ on H_1 by $\tilde{\sigma}_w(g,\mathbf{v},t) = P_1\sigma_w(g,\ P_g\mathbf{v},\ t)$, where $\sigma_w(g,P_g\mathbf{v},t)$ is a semiflow on the space H_g generated by the weak solutions of equation (2.2) with the initial condition $P_g\mathbf{v}$.

Definition 2.2. Let us define the set Λ with the metric inherited from $W^{1,\infty}(\Omega)$ as $g \in \Lambda$ if

(1) $g(\mathbf{x}) \in C^{\infty}_{per}(\Omega)$ and $\int_\Omega \frac{1}{g}\ d\mathbf{x} = 1$ with $0 < m \le g(x,y) \le M$, for all $(x,y) \in \Omega$.

(2) $\| g \|^2_{W^{1,\infty}} < \frac{m^3\pi^2}{M}$ and $\| g \|_{W^{2,\infty}} \le M_0$ for some constant M_0.

Theorem 2.3. *Let* $\mathbf{f} \in L^2(\Omega)$ *and* $g \in \Lambda \subset W^{2,\infty}(\Omega)$, *where* Λ *is given in definition 2.2. Then, for every* $g \in \Lambda$, $\tilde{\sigma}_w(g,\mathbf{v},t)$ *has a global attractor and the family of the semiflows with respect to* g, $\tilde{\sigma}_w(g,\mathbf{v},t)$, *is robust at the global attractor of the semiflow* $\tilde{\sigma}_w(1,\mathbf{v},t)$.

Also, we can define the semiflow on V_1 by $\tilde{\sigma}_s(g,\mathbf{v},t) = P_1\sigma_s(g,\ P_g\mathbf{v},\ t)$, where $\sigma_s(g,P_g\mathbf{v},t)$ is a semiflow on the space V_g generated by the strong solutions of equation (2.2) with the initial condition $P_g\mathbf{v}$. And we get the following theorem, due to Robustness theorem.

Theorem 2.4. *Let* $\mathbf{f} \in L^2(\Omega)$ *and* $g \in \Lambda \subset W^{2,\infty}(\Omega)$, *where* Λ *is given in definition 2.2. Then, for every* $g \in \Lambda$, $\tilde{\sigma}_s(g,\mathbf{v},t)$ *has a global attractor and the family of the semiflows with respect to* g, $\tilde{\sigma}_s(g,\mathbf{v},t)$, *is robust at the global attractor of the semiflow* $\tilde{\sigma}_s(1,\mathbf{v},t)$.

In next two theorems, we will see the behavior of the solutions as g goes to 1 in suitable sense.

Theorem 2.5. *Assume that* $g \in \Lambda$ *and* $\mathbf{f} \in L^2(0,\infty;L^2(\Omega,g)$ *with* $\int_\Omega \mathbf{f}\ d\mathbf{x} = 0$. *Let* $(\mathbf{v},p)$ *is the weak solution of the Navier-Stokes equations with the initial condition* $\mathbf{v}(0) = \mathbf{v}_0 \in H_1$ *and* $(\mathbf{u}_g,p_g)$ *is the weak solution of the g-Navier-Stokes equations with the initial condition* $P_g\mathbf{v}_0 = \mathbf{u}_0 = \mathbf{u}_g(0) \in H_g$. *Then, as* $g \to 1$ *in* $W^{1,\infty}(\Omega)$, *we have*

$$\mathbf{u}_g \to \mathbf{v} \quad in \quad L^2(0,T;H^1(\Omega)),\ \ in \quad L^\infty(0,T;L^2(\Omega)), \tag{2.3}$$

$$\nabla p_g \to \nabla p \quad in \quad H^{-1}(Q), \tag{2.4}$$

where $\mathcal{Q} = \Omega \times (0,T)$*, for* $0 < T < \infty$.

Theorem 2.6. *Assume that* $g \in \Lambda$ *and* $\mathbf{f} \in L^2(0,\infty; L^2(\Omega, g))$ *with* $\int_\Omega \mathbf{f}\ d\mathbf{x} = 0$*. Let* $(\mathbf{v}, p)$ *is the strong solution of the Navier-Stokes equations with the initial condition* $\mathbf{v}(0) = \mathbf{v}_0 \in V_1$ *and* $(\mathbf{u}_g, p_g)$ *is the strong solution of the g-Navier-Stokes equations with the initial condition* $P_g\mathbf{v}_0 = \mathbf{u}_0 = \mathbf{u}_g(0) \in V_g$*. Then, as* $g \to 1$ *in* $W^{2,\infty}(\Omega)$*, we have*

$$\mathbf{u}_g \to \mathbf{v} \quad in \quad L^\infty(0,T; H^1(\Omega)), \quad in \quad L^2(0,T; H^2(\Omega)) \tag{2.5}$$

$$\nabla p_g \to \nabla p \quad in \quad L^2(\mathcal{Q}), \tag{2.6}$$

where $\mathcal{Q} = \Omega \times (0,T)$*, for* $0 < T < \infty$.

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