On the regularity condition of the vorticity for the Navier-Stokes equations

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ABSTRACT

We study the local regularity criteria of "suitable" weak solutions to the Navier-Stokes equations in dimension three. We prove that if either two components of vorticiy are locally at an interior point in some Lebesgue spaces in which the norm of solutions is scaling invariant or the direction of vorticity is locally in some Triebel-Lizorkin spaces in a neighborhood of an interior point, then solutions are bounded near the point.

INTRODUCTION

In this note we study the interior regularity problem for suitable weak solutions to the Navier-Stokes equations in three dimensions with zero external force:

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v - \Delta v + \nabla p = f, \tag{1}$$

$$\text{div } v = 0 \tag{2}$$

in $Q_T := \Omega \times (0, T)$, where Ω is a domain in $\mathbb{R}^3$, v is the flow velocity, p is the scalar pressure, and $\nu > 0$ is the viscosity constant. Here it is, for simplicity, assumed that zero external force is is assigned, i.e. $f = 0$. We are especially concerned with the initial boundary value problem on a bounded and smooth domain, and therefore, we require together with (1)-(2) initial and boundary conditions:

$$v(x, 0) = v_0(x), \qquad x \in \Omega \tag{3}$$

$$v(x, t) = 0, \qquad x \in \partial\Omega, \ \ 0 < t < T. \tag{4}$$

The initial data (3) should satisfy a compatibility condition, i.e. $v_0 = 0$, $x \in \partial\Omega$ and $\operatorname{div} v_0 = 0$ in Ω. Suitable weak solutions mean, roughly speaking, functions which solve the Navier-Stokes equations in the sense of distribution, satisfy some integrability conditions, and satisfy the local energy inequality (see e.g. [4] for the details). For a point $z = (x, t)$ in $\Omega \times (0, T]$, we denote

$$B_{x,r} := \{y \in \mathbb{R}^3 : |y - x| < r\}, \quad Q_{z,r} := B_{x,r} \times (t - r^2, t).$$

A solution v is said to be *regular* at z if v is bounded in $Q_{z,r}$ for some $r > 0$.

After pioneering work of Leray[13] and Hopf[12] on the existence for the weak solutions of the Navier-Stokes equations, the regularity question of weak solutions has been considered as one of the fundamental problems in the mathematical fluid mechanics. Under an additional assumption known as Ladyzhenskaya-Prodi-Serrin's condition, that is

$$\|v\|_{L^{q,p}_{t,x}(Q_T)} := \Big\| \|v(\cdot,t)\|_{L^p_x(\Omega)} \Big\|_{L^q_t(0,T)} < \infty, \quad \frac{3}{p}+\frac{2}{q} \leq 1, \quad 3 \leq p \leq \infty, \tag{5}$$

it was proved that any weak solution becomes unique and regular in Q_T. An opening result in this direction was done by Serrin [15] in case that $3/p+2/q<1$ and $3<p\leq\infty$. Later, Fabes, Jones, and Riviére [10] extended Serrin's result to the limiting case $3/p+2/q=1$ and $3<p\leq\infty$ for $\Omega=\mathbb{R}^n$ (see also Sohr [16] and Giga [11] for a domain). For the local problem, the interior case was proved by Struwe [17]. The borderline case $p=3$ and $q=\infty$ was recently solved by Escauriaza, Seregin, and Šverák [9] (there are many other achievements and developments, but we will not try to give a list here).

On the other hand, there have been works on regularity criteria concerning the gradient of velocity or vorticity rather than the velocity. Taking curl of the momentum equations (1), we obtain the following vorticity equations:

$$\frac{\partial \omega}{\partial t} + (v\cdot\nabla)\omega - \Delta\omega = (\omega\cdot\nabla)v, \tag{6}$$

where the vorticity ω is defined by $\omega = \text{curl } v$. Taking the incompressibility of v into account, due to the Biot-Savart's law, v can be represented as the form of a singular integral of ω, namely,

$$v(x,t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{y\times\omega(x+y,t)}{|y|^3}dy \tag{7}$$

for sufficiently rapidly decaying vorticity near infinity. Beirão da Veiga [2] obtained a sufficient condition for regularity in terms of ∇v instead of the velocity. More preciely, in [2], it was proved that if ∇v satisfies

$$\nabla v \in L^q(0,T;L^p(\mathbb{R}^3)), \quad \frac{3}{p}+\frac{2}{q}\leq 2, \quad \frac{3}{2}<p\leq\infty, \tag{8}$$

then v is regular. This result was improved in [6], imposing the condition (9) below only for the two components of the vorticity, more precisely, let $\omega=\tilde{\omega}+\omega'$, where $\tilde{\omega}=\omega_1 e_1+\omega_2 e_2$ and $\omega'=\omega_3 e_3$ ($e_i, i=1,2,3$ are standard basis for $\mathbb{R}^3$). If $\tilde{\omega}$ satisfies

$$\tilde{\omega} \in L^q(0,T;L^p(\mathbb{R}^3)), \quad \frac{3}{p}+\frac{2}{q}\leq 2, \quad \frac{3}{2}<p<\infty, \tag{9}$$

then v becomes a classical solution (see [6, Theorem 1]).

One of our main results is to establish the local version of the regularity criteria (9) for the interior case. We have the following theorem for the local problem in the interior.

Theorem 1 *Let $z_0=(x_0,t_0)\in Q_T, Q_{z_0,r}\Subset Q_T$, and $e_i, i=1,2,3$ be the standard basis for $\mathbb{R}^3$. If v is a suitable weak solution of the Navier-Stokes equations* (1)-(4) *and $\tilde{\omega}:=\omega-(\omega\cdot e_3)e_3$, the two components of the vorticity ω, satisfy*

$$\tilde{\omega}\in L^{q,p}_{t,x}(Q_{z_0,r}), \quad \frac{3}{p}+\frac{2}{q}\leq 2, \quad \frac{3}{2}<p<\infty, \tag{10}$$

then z_0 is a regular point.

PROOF. See the proof of Theorem 1 in [7] for the details.

We remark that for the case $p = 3/2$ and $q = \infty$, if we add the smallness condition on $\tilde{\omega}$, i.e., $\|\tilde{\omega}\|_{L^{\infty,3/2}_{t,x}(Q_{z_0,r})} < \epsilon$ for a sufficiently small $\epsilon > 0$, then z_0 is a regular point.

On the other hand, Constantin and Fefferman[8] discovered sufficient conditions on $\xi(x,t) = \omega(x,t)/|\omega(x,t)|$, the direction of the vorticity, for the existence of a smooth solution of the Navier-Stokes equations in $\mathbb{R}^3$. They proved that if $\theta(x, y.t)$, the angle between $\xi(x,t)$ and $\xi(x+y,t)$, satisfies

$$|\sin\theta(x,y,t)| \le C|y|, \tag{11}$$

then the solution becomes regular. In [3], the condition (11) was relaxed in the form of the following regularity criteria: there exist $s \in [1/2, 1]$, a constant $K > 0$ and $g \in L^r(0,T; L^p(\mathbb{R}^3))$ where

$$\frac{3}{p} + \frac{2}{q} = s - \frac{1}{2}, \quad q \in [\frac{4}{2s-1}, \infty],$$

such that $|\sin\theta(x,y,t)| \le g(x,t)|y|^s$ in a region where the vorticity at both x and $x+y$ is larger than K. Under this assumption, it was proved that if v is a weak solution of the Navier-Stokes equations, then the solution is regular (see [3, Theorem 1.2]). Moreover, for the case $s \in (0, 1/2)$, one additional condition on the integrability of vorticity together with its direction ensures the regularity of weak solutions (see [1, Theorem 1.1]). The first author [5] recently unified and refined the results above initiated by Constantin and Fefferman, measuring direction fields in terms of the norms of Triebel-Lizorkin type. The main result in [5] is as follows: There exist $s \in (0,1)$, $q \in (3/(3-s), \infty]$, $p_1 \in (1, \infty]$, $p_2 \in (1, 3/s)$ and $r_1, r_2 \in [1, \infty]$ satisfying

$$\frac{3}{p_1} + \frac{3}{p_2} + \frac{2}{r_1} + \frac{2}{r_2} \le 2 + s, \quad \frac{s}{3} < \frac{1}{p_1} + \frac{1}{p_2} < \frac{2+s}{3}, \quad \frac{1}{p_2} + \frac{1}{q} < 1 + \frac{s}{3}, \tag{12}$$

such that if

$$\xi(x,t) \in L^{r_1}(0,T; \dot{\mathcal{F}}^s_{p_1,q}), \quad \omega(x,t) \in L^{r_2}(0,T; L^{p_2}(\mathbb{R}^3)), \tag{13}$$

then there is no singularity up to time T. Here $\dot{\mathcal{F}}^s_{p,q}$ indicates Triebel-Lizorkin type function space (see [5] and [18] for the details). Our second result is that the sufficient condition for regularity in [5] can be localized in the interior.

Theorem 2 *Let $z_0 = (x_0, t_0) \in Q_T$ and $Q_{z_0,r} \Subset Q_T$. Suppose v be a suitable weak solution of the Navier-Stokes equations* (1)-(4), *and $\omega = \operatorname{curl} v$. Let $\xi(x,t)$ be the directional field $\xi = \omega(x,t)/|\omega(x,t)|$ defined for $\omega(x,t) \ne 0$. There exist $s \in (0,1)$, $q \in (\frac{3}{3-s}, \infty]$, $p_1 \in (1,\infty]$, $p_2 \in (1, \frac{3}{s})$, and $r_1, r_2 \in [1,\infty]$ such that if*

$$\frac{s}{3} < \frac{1}{p_1} + \frac{1}{p_2} < \frac{2+s}{3}, \quad \frac{1}{p_2} + \frac{1}{q} < 1 + \frac{s}{3}, \quad \frac{3}{p_1} + \frac{3}{p_2} + \frac{2}{r_1} + \frac{2}{r_2} \le 2 + s,$$

and

$$\xi(x,t) \in L^{r_1}(t_0 - r^2, t_0; \dot{\mathcal{F}}^s_{p_1,q}(B_{x_0,r})), \quad \omega(x,t) \in L^{r_2,p_2}_{t,x}(Q_{z_0,r}), \tag{14}$$

then z_0 is a regular point.

PROOF. See the proof of Theorem 2 in [7] for the details.

In the above Theorem 2, $\dot{\mathcal{F}}^s_{p_1,q}(B_{x_0,r})$ indicates local homogeneous Triebel-Lizorkin type function spaces (see [7, Definition 4] for the details). We also remark that, since we know $\omega \in L^2(Q_{z_0,r})$ from the weak formulation of solutions, the second condition in (14) is unnecessary provided that $s \geq 1/2$, and therefore, we can impose the assumption only on the directional field $\xi(x,t)$ for such case, i.e., $\xi \in L^{r_1}(t_0 - r^2, t_0; \dot{\mathcal{F}}^s_{p_1,q}(B_{x_0,r}))$ satisfying $3/p_1 + 2/r_1 \leq s - 1/2$ in Theorem 2.

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