PRESSURE REPRESENTATION WITH SLIP BOUNDARY CONDITION

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ABSTRACT

We first represent the pressure in terms of the velocity in $\mathbb{R}^3_+$. Using this representation we prove that a solution to the Navier-Stokes equations is in $L^\infty(\mathbb{R}^3_+ \times (0,\infty))$ under the critical assumption that $u \in L^{r,r'}_{loc}$, $\frac{3}{r}+\frac{2}{r'} \le 1$ with $r \ge 3$. In [4], a boundary L^∞ estimate for the solution is derived if the pressure on the boundary is bounded. In our work, we remove the boundedness assumption of the pressure. Here, our estimate is local. Indeed, employing Moser type iteration and the reverse Hölder inequality, we find an integral estimate for L^∞-norm of $\mathbf{u}$.

INTRODUCTION AND STATEMENT OF THE RESULT

In the Navier-Stokes problem, the pressure is most troublesome to find a regularity estimate. To avoid the indirect introduction of the pressure in the formulation, the suitable weak solution is introduced by Scheffer [14,15] and Caffarelli-Kohn-Nirenberg [3]. For instance, in the definition of a suitable weak solution of the Navier-Stokes equations of [3] on an open set $D = \Omega \times (0,T) \subset \mathbb{R}^3 \times \mathbb{R}$, the condition on the pressure p is that $p \in L^{5/4}(D)$. When $\Omega = \mathbb{R}^3$, the pressure satisfies $p \in L^{5/3}(D)$ by Calderon-Zygmund estimate with the parabolic Sobolev embedding. In [3], they assumed $p \in L^{5/4}(D)$ because that is the best L^q estimate known for the initial boundary value problem in the case that the domain Ω is bounded. But, with more careful study in the context of semigroup, Sohr and von Wahl [17] showed $p \in L^{5/3}(D)$ for a bounded or exterior domain Ω, too. Based on this estimate, Lin [12] found a blow up argument of the proof of the partial regularity in [3] which shortened the original proof of Caffarelli-Kohn-Nirenberg.

In view of partial regularity, there still remain several issues near boundary and again the difficulty lies on the pressure estimate. In this work, we represent the pressure by the velocity in the half space $\mathbb{R}^3_+$. As a matter of fact, it is well known that $\|p\|_{L^q(\mathbb{R}^3)} \le \|\mathbf{u}\|^2_{L^{2q}(\mathbb{R}^3)}$ for $1 < q < \infty$. Our representation also implies that $\|p\|_{L^q(\mathbb{R}^3_+)} \le \|\mathbf{u}\|^2_{L^{2q}(\mathbb{R}^3_+)}$ for $1 < q < \infty$. As an application of the representation, we show the boundary regularity of weak solutions in $\mathbb{R}^3_+ \times (0,T)$ under Serrin conditions near boundary.

In this paper we study the incompressible Navier-Stokes equations with viscosity ν

$$\partial_t u_i - \nu\Delta u_i + (u\cdot\nabla)u_i + \partial_{x_i} p = f_i, \qquad \nabla\cdot u = 0 \tag{1}$$

in $D = \mathbb{R}^3_+ \times (0,\infty)$ with the initial data, and the slip boundary condition(BC)

$$\mathbf{u}(x,0) = \mathbf{u}_0(x) \in L^2(\mathbb{R}^3_+) \qquad \text{for } x \in \mathbb{R}^3_+, \tag{2}$$
$$\partial_3 u_1(x,t) = \partial_3 u_2(x,t) = u_3(x,t) = 0 \quad \text{ for } x_3 = 0, t \in (0,\infty), \tag{3}$$
$$\mathbf{u}(\mathbf{x}) \to 0 \text{ as } |\mathbf{x}| \to \infty, \tag{4}$$

where $\Omega = \mathbb{R}^3_+$ is the half space $\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$. We let the initial data $\mathbf{u}_0$ satisfy $\nabla \cdot \mathbf{u}_0 = 0$ in Ω and (3)-(4) in a weak sense. We assume that any weak solution $\mathbf{u} \in L^2(0,\infty; H^1(\Omega)) \cap L^\infty(0,\infty; L^2(\Omega))$ satisfies

$$\int \mathbf{u} \cdot \phi_t - \nabla \mathbf{u} \cdot \nabla \phi - (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \phi + p \nabla \cdot \phi - \mathbf{f} \cdot \phi dz = 0$$

for all $\phi \in C_0^\infty(D)$. This problem corresponds to the free surface problem for the Navier-Stokes equations, for details refer to Maremonti [13], Itoh and Tani [9]. As matter of fact, the existence of the weak or strong solutions is studied by many authors, for example [13], [9], and Solonikov and Ščadilov [18]. The existence of weak solutions with the non-slip boundary condition ($\mathbf{u} = 0$ on $\partial\Omega$) was proved by Leray [11] and Hopf [8], and the existence of suitably weak solutions was proved by Caffarelli, Kohn and Nirenberg [3]. Here, our definition of a weak solution coincides with a similar version of the definition of the suitably weak solution of [3]. Since the viscosity can be treated by scaling, we simply assume that $\nu = 1$. Also, for the simplicity we assume that $\mathbf{f}$ is a smooth function in $\overline{D}$ and $\nabla \cdot \mathbf{f} = 0$.

It is well known that for a smooth domain Ω which is bounded or unbounded, if the viscosity is large or data are small, then the weak solution with the non-slip BC lies in $L^\infty(0,\infty : H^1(\Omega)) \cap L^2(0,\infty : H^2(\Omega))$. Also if the solution is bounded, then it lies in $L^\infty(0,\infty : H^1(\Omega)) \cap L^2(0,\infty : H^2(\Omega))$. We know that boundedness of $\mathbf{u}$ implies higher regularity of $\mathbf{u}$ in the interior and hence we can bound various higher norms in terms of L^∞-norm of $\mathbf{u}$. From Sobolev's embedding theorem we know that the solution space of weak solution $L^2(0,\infty; H^1(\Omega)) \cap L^\infty(0,\infty; L^2(\Omega))$ lies in $L^{\frac{10}{3}}_{loc}(D)$. But we do not know yet how to bound L^∞-norm of $\mathbf{u}$ in terms of $L^{\frac{10}{3}}$-norm of $\mathbf{u}$. On the other hand as far as interior is concerned, it was proved by Serrin [16] that any weak solution $\mathbf{u}$ of (1) on a cylinder $B \times (a,b)$ satisfying

$$\int_a^b \Big(\int_B |\mathbf{u}|^r \, dx \Big)^{\frac{r'}{r}} dt < \infty \quad \text{with} \quad \frac{3}{r} + \frac{2}{r'} < 1, \; r \geq 3$$

is necessarily L^∞ function on any compact subsets of the cylinder. Observe that when $r = r' = 5$, then $\mathbf{u}$ is in L^5 and 5 is the critical number for the homogeneous Lebesgue spaces. The limiting case $3/r + 2/r' = 1, r > 3$ for the initial value problem was proved by Fabes-Jones-Riviere [6] and their method seems not applicable to local problems. Also Struwe [19] improved Serrin's method and proved the boundedness of weak solutions in the interior for the critical case, that is, $\frac{3}{r} + \frac{2}{r'} = 1, r > 3$. Quite recently, Escauriaza, Seregin and Sverak showed that $\mathbf{u} \in L^{3,\infty}$ implies $\mathbf{u}$ is regular. Takahashi [20] found some criterion for L^∞ regularity near boundary for the weak solution satisfying $\mathbf{u} \in L^{r,r'}$, $\frac{3}{r} + \frac{2}{r'} \leq 1$. He imposed some integrability conditions on the velocity gradient and pressure in the domain D, that is,

$$\nabla \mathbf{u}, p \in L^{r_0, r_0'} \quad \text{for all} \quad 1 < r_0, r_0' < \infty \quad \text{with} \quad \frac{3}{r_0} + \frac{2}{r_0'} = 3.$$

Recently, Kang [10] showed that for $\Omega = \mathbb{R}^3_+$, a weak solution $\mathbf{u}$ is Hölder continuous up to boundary when $\mathbf{u}$ belongs locally to $L^{r,r'}$ at a boundary point where $\frac{3}{r} + \frac{2}{r'} = 1$, $r > 3$.

The second author [4] have shown that the L^∞ boundary regularity of $\mathbf{u}$ with non-slip BC up to boundary for the limiting case that $\mathbf{u} \in L^{r,r'}(D)$, $\frac{3}{r} + \frac{2}{r'} \le 1$ with $r > 3$ or $\mathbf{u} \in L^{3,\infty}$ with $\|\mathbf{u}\|_{L^{3,\infty}} \le \varepsilon_0$ for some small ε_0 under the assumption that the boundary data of the pressure is bounded. For the proof, it is shown that $\mathbf{u} \in L^5$, if $u \in L^{r,r'}(D)$, $\frac{3}{r} + \frac{2}{r'} \le 1$ with $r > 3$ or $\mathbf{u} \in L^{3,\infty}$ with $\|\mathbf{u}\|_{L^{3,\infty}} \le \varepsilon_0$ for some small ε_0. Employing Moser type iteration, it is shown that $\|\mathbf{u}\|_\infty$ can be bounded by $\|\mathbf{u}\|_p$ for all $p > 5$. Then from the reverse Hölder inequality $\|\mathbf{u}\|_{5+\sigma}$ for some σ can be bounded by $\|\mathbf{u}\|_5$. Combining these two estimates, $\|\mathbf{u}\|_\infty$ is bounded in terms of $\|\mathbf{u}\|_5$. In the same paper it is also shown that the weak solution is as regular as the boundary data of the pressure. It is also proved that if the tangential derivatives of the pressure is bounded, the velocity $\mathbf{u}$ is $C^{1,\alpha}$ continuous as a function of space variables. This is achieved from a comparison of the weak solution and caloric function and the fact that caloric functions satisfy Campanato type integral inequality. This method has been well established in the regularity theory of parabolic systems and elliptic systems(see [3] and [4]). Then, the $C^{1,\alpha}$ regularity will follows from the isomorphism theorem of Campanato space and Hölder space. From a bootstrap argument the weak solution $\mathbf{u}$ will be as regular as the boundary data of the pressure.

This paper consists of three parts. The first part, which is our main result, is about pressure representation. We represent the pressure in terms of the velocity in $\mathbb{R}^3_+$, which implies that $\|p\|_{L^q(\mathbb{R}^3_+)} \le \|\mathbf{u}\|^2_{L^{2q}(\mathbb{R}^3_+)}$ for $1 < q < \infty$ likewise in $\mathbb{R}^3$.

Theorem 1 *Suppose* $\mathbf{u}, p$ *are a measurable function and a distribution, respectively, satisfying* (1)-(4) *in the sense of distributions. Then* p *has the following representation; for almost all time* $t \in (0, T)$

$$p(\mathbf{x}, t) = \frac{-\delta_{ij}}{3}(u_i^* u_j^*)(\mathbf{x}, t) + \frac{3}{4\pi}\int_{\mathbb{R}^3} \left(\frac{\partial^2}{\partial y_i \partial y_j}\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)(u_i^* u_j^*)(\mathbf{y}, t)\, d\mathbf{y}, \tag{5}$$

where δ_{ij} *is the Kronecker delta function. Here,* $\mathbf{u}^*(\mathbf{y}) = \mathbf{u}(\mathbf{y})$ *for* $y_3 > 0$*, and*

$$u_1^*(\mathbf{y}, t) = u_1(\mathbf{y}^*, t), \quad u_2^*(\mathbf{y}, t) = u_2(\mathbf{y}^*, t), \quad u_3^*(\mathbf{y}, t) = -u_3(\mathbf{y}^*, t)$$

for $y_3 < 0$*, and* $\mathbf{y}^* = (y_1, y_2, -y_3)$.

The second is about boundary regularity, which is an application of our pressure representation. We provide a simple proof of L^∞ boundedness of $\mathbf{u}$ up to boundary for the case that $\mathbf{u} \in L^{r,r'}$, $\frac{3}{r} + \frac{2}{r'} \le 1$ with $r \ge 3$ without any restriction on the boundary data of the pressure. In case $r = 3$, we need the smallness of $\mathbf{u}$.

Theorem 2 *Suppose* $(\mathbf{u}, p)$ *is a weak solution. There exists a positive constant* σ *such that if* $\mathbf{u} \in L^{r,r'}(Q_2^+)$ *for some* (r, r') *satisfying* $\frac{3}{r} + \frac{2}{r'} \le 1$ *with* $r > 3$*, then*

$$\sup_{Q^+_{\frac{1}{8}}} |\mathbf{u}| \le c\left(\int_{Q_1^+} |\mathbf{u}|^3\, dz\right)^{\frac{5+\sigma}{3\sigma}} + c$$

for some positive constant c *depending on* ε_0.

Finally, we provide the existence and the definition of the suitable weak solutions in the appendix.

Using the representation, we also showed the only self similar solution in $\mathbb{R}^3_+ \times (0, T)$ is zero in our another paper [1].

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