# On $L^p$ Estimates for Nonlinear Equations

Sun-Sig Byun [1] and Lihe Wang [2]

1) *Department of Mathematical Sciences, Seoul National University, Seoul 151-742, KOREA*

2) *Department of Mathematics, University of Iowa, IA 52242, USA*

Corresponding Author : Sun-Sig Byun, byun@math.snu.ac.kr

**ABSTRACT**

We will study in this note a global $W^{1,p}$ estimate for $1 < p < \infty$ for the weak solution to a boundary-value problem for a divergence structure nonlinear elliptic partial differential equation that is not of variational form. The boundary of a domain is only assumed to be nontangentially accessible. We will ask what are reasonable conditions to place both on the nonlinearity and on the boundary for the $W^{1,p}$ regularity to this large class of elliptic problems for many interesting domains with fractal boundaries.

## INTRODUCTION

We study a global estimate for the gradient of a weak solution to a divergence structure nonlinear elliptic partial differential equation that is not of variational form in an irregular domain. In particular we are interested in the well-posedness of the following nonlinear boundary-value problem

$$\begin{cases} -\mathrm{div}\,\mathbf{a}(\nabla u, x) = \mathrm{div}\,\mathbf{f} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

Here $\mathbf{f} \in L^p(\Omega)$ is a given vector field for some $p > 1$, as is the smooth vector field $\mathbf{a} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\mathbf{a} = (a^1, ..., a^n)$. The unknown is $u : \overline{\Omega} \longrightarrow \mathbb{R}$, $u = u(x)$, where $\Omega$ is a bounded, open subset of $\mathbb{R}^n$. As usual, solutions of (1) are taken in a weak sense. We use the following classical definition of a weak solution.

**Definition 1** *We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of (1) if we have*

$$\int_\Omega \mathbf{a}(\nabla u, x) \cdot \nabla\varphi \, dx = -\int_\Omega \mathbf{f} \cdot \nabla\varphi \, dx$$

*for all $\varphi \in C_0^\infty(\Omega)$.*

We refer to [6] for a general discussion on the weak solution of (1). Now if there exists a function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $\mathbf{a}(\xi, x)$ is the gradient of $F(\xi, x)$ with respect to $\xi \in \mathbb{R}^n$, then (1) is the Euler-Lagrange equation corresponding to the variational integral

$$I[w] = \int_\Omega \left(F(\nabla w, x) + \mathbf{f} \cdot \nabla w\right) \, dx \, \left(w \in W_0^{1,p}\right).$$

However, if the PDE (1) is not of variational type, i.e., there exists no such potential $F$, the classical variational methods do not apply to our present problem for the existence of a weak solution. We use instead monotonicity methods. To do this let us assume that $\mathbf{a}$ is uniformly monotonic in $\xi$, i.e., there exist some positive constants $\lambda$, $\Lambda$ such that

$$\lambda\,|\xi-\eta|^2 \le (\mathbf{a}(\xi,x)-\mathbf{a}(\eta,x))\cdot(\xi-\eta) \le \Lambda\,|\xi-\eta|^2 \tag{2}$$

for all $\xi,\ \eta \in \mathbb{R}^n$ and almost every $x \in \Omega$, and that

$$|\mathbf{a}(\xi,x)| \le C(1+|\xi|^{p-1}),$$

$$\mathbf{a}(\xi,x)\cdot\xi \ge a|\xi|^p - b$$

for all $\xi \in \mathbb{R}^n$ and some constants $C$, $a>0$ and $b\ge 0$.

In this work we are interested in studying how the regularity of $\mathbf{f}$ is reflected to the solutions under minimal assumptions on the vector field $\mathbf{a}$ and the smoothness requirement on the domain $\Omega$. More precisely, we want to ask what are optimal conditions to place both on the nonlinearity $\mathbf{a}$ and on the boundary $\partial\Omega$ under which we have the following $W^{1,p}$ estimates:

$$\int_\Omega |\nabla u|^p\,dx \le C\int_\Omega |\mathbf{f}|^p\,dx,$$

where the constant $C$ is independent of $u$ and $\mathbf{f}$. First we point out that the $L^p$ regularity we discuss here is not of the kind of classical one (i.e. valid for $p=2$, see [6]). On the contrary our results hold for any value of $p$ in the range $(1,+\infty)$. Obviously, such a result requires additional conditions on $\mathbf{a}$ and $\partial\Omega$.

In the uniformly elliptic, second-order PDE case, Di Fazio in [5] proved a global $W^{1,p}(\Omega)$ estimate for $1<p<\infty$ estimate provided the coefficients are bounded functions of vanishing mean oscillation and the domain is in $C^{1,1}$. His argument is based on explicit representation formulas involving singular integral operators and commutators, and works only for the linear equations in this direction. The authors in [1] obtained the same result as in [5] under much weaker assumptions, which are that the coefficients have a suitable smallness condition in the John-Nirenberg space BMO of the functions of bounded mean oscillation (see [8]), and that the domain is sufficiently flat in the Reifenberg sense. Roughly speaking, a set is flat in the Reifenberg sense if it is well approximated by hyperplanes at every point and at every scale (see [10,11]). Our approach relies on weak compactness, the Hardy-Littlewood maximal function, the Vitali covering lemma, good $\Lambda$-inequalities, and energy estimates. Recently the authors in [2,3] extend the results in [1] to elliptic PDEs of $p$-Laplacian type that is of variation form. In this note we attack a divergence structure nonlinear elliptic PDE that is not of variational form by combining the approaches used in [1,4]. We will develop a unifying method to obtain the interior and boundary estimates for the highly nonlinear case. This method is more flexible to deal with the irregular domains than the flattening argument since the latter always requires some differentiability conditions on the boundaries of the domains. The main difficulty in our work comes from the nonlinearity of the equation (1), which causes the counterparts in nonlinear case for the obvious conclusions in linear case to be hard to prove. This work is a natural outgrowth of the papers [2,3], where the similar results are obtained for elliptic equations of variational type. Although we use almost the same functional framework and analytic procedure as in [1–3], more complicated analysis has to be carefully carried out with great patience. This could be an intriguing development since the present tools are quite flexible and suitable for both linear and nonlinear problems. In particular, these techniques can be used in many other situations like subelliptic and parabolic operators. In a forthcoming paper we will extend the present result to a parabolic setting.

## APPROXIMATION PROPERTY HYPOTHESES

In this section we describe the hypotheses on the vector field $\mathbf{a}$ and on the domain $\Omega$. Let us begin with the statement of the general assumption on $\mathbf{a}$. In order to measure the oscillation of $\mathbf{a}(\xi, x)$ in the variable $x$ over each ball $B_r(y) \subset \mathbb{R}^n$, we consider the function

$$\beta(x) = \sup_{\xi \in \mathbb{R}^n} \frac{\left| \mathbf{a}(\xi, x) - \overline{\mathbf{a}(\xi, .)}_{B_r(y)}) \right|}{|\xi| + 1},$$

where $\overline{\mathbf{a}(\xi, .)}_{B_r(y)}$ is the integral average of $\mathbf{a}(\xi, .)$ for each fixed $\xi$ over the balls $B_r(y)$. Motivated the early works in [1–4] we use the following assumption.

**Definition 2** *We say that the vector field* $\mathbf{a}$ *is* $(\delta, R)$*-vanishing if*

$$\sup_{0<r\leq R} \sup_{y \in \mathbb{R}^n} \frac{1}{|B_r|} \int_{B_r(y)} |\beta(x)| \, dx \leq \delta. \tag{3}$$

Our geometric setting for the domain is stated as the following.

**Definition 3** *Let* $0 < \delta \leq R$*. Then we say that* $\Omega$ *is* $(\delta, R)$*-Reifenberg flat if every* $x \in \partial\Omega$ *and every* $r \in (0, R]$*, there exists a coordinate system* $\{y_1, \ldots, y_n\}$*, which can depend on* $r$ *and* $x$ *so that* $x = 0$ *in this coordinate system and that*

$$B_r(0) \cap \{y_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y_n > -\delta r\}.$$

In the definitions above we mean $\delta$ to be a small positive constant while one can assume $R = 1$ by a scaling. A Reifenberg flat domain was introduced by Reifenberg in the paper [11] where he proved that it is locally a topological disk if $\delta$ is sufficiently small. A typical example of Reifenberg flat domains is the well-known Van Koch snowflake. The Van Koch curve is a self-similar Jordan curve and a prototypical fractal set. Fractals are geometric shapes that are very complex and infinitely detailed. We can zoom in on a section and it will have just as much detail as the whole fractal. They are recursively defined and small sections of them are similar to large ones. They are found in real-world systems such as blood vessels, the internal structure of the lungs, graphs of stock market data, bacteria and fern growth, clouds, mountains and so on. The remarkable thing for Reifenberg flat domains is that they are $W^{1,p}$ extension domains (see [7,9]). Thus extension theorem and Sobolev inequalities are available on a $(\delta, R)$-Reifenberg flat domain, which is very important to the $W^{1,p}$ regularity theory for elliptic equations as well as parabolic equations.

## MAIN THEOREM

The main theorem in this note is the following.

**Theorem 4** *Let* $p > 1$ *be a real number. Then there is a small* $\delta = \delta(\lambda, \Lambda, p, n, R) > 0$ *so that for all* $A$ *with* $\mathbf{a}$ $(\delta, R)$*-vanishing, for all* $\Omega$ *with* $\Omega$ $(\delta, R)$*-Reifenberg flat, and for all* $\mathbf{f}$ *with* $\mathbf{f} \in L^p(\Omega; \mathbb{R}^n)$*, the Dirichlet problem (1) has a unique weak solution with the estimate*

$$\int_\Omega |\nabla u|^p dx \leq C \int_\Omega |\mathbf{f}|^p dx,$$

*where the constant* $C$ *is independent of* $u$ *and* $\mathbf{f}$*.*

## REFERENCES


1. Byun, S. and Wang, L., “Elliptic equations with BMO coefficients in Reifenberg domains”, *Comm. Pure Appl. Math.*, 2004, pp. 1283-1310.
2. Byun, S. and Wang, L., “Quasilinear Elliptic equations with BMO coefficients in Lipschitz domains”, *Trans. Amer. Math. Soc.*, to appear.
3. Byun, S., Wang, L. and Zhou S., “Quasilinear Elliptic equations with BMO coefficients in Reifenbereg domains”, submitted
4. Caffarelli L.A. and Peral I., “On $W^{1,p}$ estimates for elliptic equations in divergence form”, *Comm. Pure Appl. Math.*, 1998, pp. 1-21.
5. Di Fazio, G., “$L^p$ estimates for divergence form elliptic equations with discontinuous coefficients”, *Boll. Un. Mat. Ital A(7)*, 1998, 409-420.
6. Evans, Lawrence C., *Partial Differential Equations*, Graduate Studies in Mathematics, **19**, American Mathematical Society, Providence, RI, 1998.
7. Hajlasz, P. and Martio, O., “Traces of Sobolev functions on fractal type sets and characterization of extension domains”, *J. Funct. Anal.*, 1997, 221-246.
8. John, F. and Nirenberg, L., “On functions of bounded mean oscillation”, *Comm. Pure Appl. Math.*, 1961, 415-426.
9. Jones, P.W., “Quasiconformal mappings and extendability of functions in Sobolev spaces”, *Acta Math.*, 1981, 71-88.
10. Kenig, C. and Toro, T., “Free boundary regularity for harmonic measures and the Poisson kernel”, *Ann. of Math.*, 1999, 367-454.
11. Reinfenberg, E., “Solutions of the plateau problem for m-dimensional surfaces of varying topological type”, *Acta Math.*, 1960, 1-92.