THE EVALUATION OF SCATTERED POINTS QUADRATURE RULES

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ABSTRACT

For a given set of scattered points $\mathbf{X} := \{\mathbf{s}_j\}_{j=0}^M$ in I, we will establish a quadrature formula on I, which is exact for polynomials of degree less than or equal to K by using $(M+1)$ scattered points with $(M+1) \geq \dim \mathcal{P}_K(I)$. To this end, we first define the 'density' of $\mathbf{X}$ in $\overline{\Omega}$ to be the number

$$h := h(X;\Omega) := \sup_{\mathbf{t}\in\Omega} \min_{\mathbf{s}_j \in \mathbf{X}} |\mathbf{t} - \mathbf{s}_j| \tag{1}$$

and introduce the notion of 'admissible vector' $(a(\cdot, \mathbf{s}_j))_{j=0}^M$.

Definition 0.1 *The vector $(a(\cdot, \mathbf{s}_j))_{j=0}^M$ are termed admissible for $\mathcal{P}_n(\Omega)$ if they satisfy the following three conditions:*

(a) There exists a positive constant $c > 0$ such that, for any $\mathbf{t} \in \Omega$, $a(\mathbf{t}, \mathbf{s}_j) = 0$ whenever $|\mathbf{t} - \mathbf{s}_j| > ch$, with h the density of $\mathbf{X}$ as in (1).

(b) For every $\mathbf{t} \in I$, $(a(\mathbf{t}, \mathbf{s}_j))_{j=0}^M$ reproduces polynomials in $\mathcal{P}_n(\Omega)$, i.e.,

$$\sum_{j=0}^M a(\mathbf{t}, \mathbf{s}_j) p(\mathbf{s}_j) = p(\mathbf{t}), \quad \forall p \in \mathcal{P}_n(\Omega). \tag{2}$$

The key point in this numerical quadrature is to find a unique admissible vector $\Big(a(\mathbf{t}, \mathbf{s}_l)\Big)_{l=0}^M$ such that for all $p \in \mathcal{P}_K(I)$,

$$\sum_{l=0}^M p(\mathbf{s}_l) a(\mathbf{t}, \mathbf{s}_l) = p(\mathbf{t}), \tag{3}$$

so that

$$\int_\Omega p(\mathbf{t}) w(\mathbf{t})\, d\mathbf{t} = \sum_{j=0}^M p(\mathbf{s}_j)\hat{w}_j \quad \forall p \in \mathcal{P}_n(I) \tag{4}$$

for some suitable $\hat{w}_j$, $j = 0, \cdots, M$. The above system (4) is undetermined. For the basis of $\mathcal{P}_N(I)$, we will denote it as $\{\phi_i\}_{i=k}^{N+k}$ which is the set of local Lagrange polynomials of degree N with respect to representation grid $\{\mathbf{s}_i\}_{i=k}^{N+k}$ in I where

$$0 \leq k < N + k \leq M,$$

which satisfy

$$\phi_i(\mathbf{s}_j) = \delta_{ij}, \quad \forall\, i, j = k, \cdots, N+k$$

where δ_{ij} denotes the Kronecker delta. In fact, it can be uniquely determined by making the kernel $a(t, s_l)$ in (3) be defined locally Lagrange fundamental Polynomial. The next lemma treats how the weight $\hat{w}_j$, $j = 0, \cdots, M$, are constructed.

Lemma 0.1 *Let $w(\mathbf{t})$ be a weight function and assume that the vector $(a(\cdot,\, \mathbf{s}_j))_{j=0}^{M}$ are admissible for $\mathcal{P}_n(I)$. Then, for any polynomial $p \in \mathcal{P}_n(I)$, we have*

$$\int_\Omega p(\mathbf{t})w(\mathbf{t})d\mathbf{t} = \sum_{j=0}^{M} p(\mathbf{s}_j)\hat{w}_j \tag{5}$$

with $\hat{w}_j$ of the form

$$\hat{w}_j := \int_\Omega a(\mathbf{t},\, \mathbf{s}_j)w(\mathbf{t})\, d\mathbf{t}, \quad j = 0, \cdots, M. \tag{6}$$

Now, we estimates the error of the quadrature in (5) and (6) to the functions in the Sobolev space.

Theorem 0.1 *Let $w(\mathbf{t})$ be a weight function and $\hat{w}_j$, $j = 0, \cdots, M$, be the quadrature in (5). If $f \in H_p^k(\Omega)$, then we get*

$$\int_\Omega f(\mathbf{t})w(\mathbf{t})d\mathbf{t} = \sum_{j=0}^{M} \hat{w}_j f(\mathbf{s}_j) + E_M(f)$$

with

$$E_M(f) := \sum_{|\alpha|_1 = n+1} \sum_{j=0}^{M} \int_\Omega a(\mathbf{t},\, \mathbf{s}_j)w(\mathbf{t}) \frac{(\mathbf{t}-\mathbf{s}_j)^\alpha}{\alpha!} \int_0^1 (n+1)(1-y)^n D^n f(\mathbf{t}+y\mathbf{s}_j)\, dy d\mathbf{t}.$$

This quadrture formula will be turned out same as, so called, Clenshaw-Curtis quadrature rule([1]) when it is restricted to the interval $[-1, 1]$, CGL nodes, and $M = K$ are chosen in $1D$ case. This observation yields that the new quadrature rule (5) and (6) is considered as an extension of Clenshaw-Curtis rule. Also, we will propose useful quadrature rules for some kinds of points set.

REFERENCES

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