

# A Finite Element Gauge-Uzawa Method for the Evolution Navier-Stokes Equations

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## ABSTRACT

The Navier-Stokes of incompressible fluids are still a computational challenge. The numerical difficulty arises from coupling variables for velocity and pressure, which requires a compatibility condition (discrete inf-sup) between the finite element spaces. Several projection methods [2,5,7] have been introduced for time discretization to circumvent the incompressibility constraint, but suffer from boundary layers. They are either numerical or due to non-physical boundary conditions on pressure.

We introduce a first order gauge-Uzawa method in [1,3,5] for time discretization coupled with a stable finite elements method for space discretization. The method is unconditionally stable, consists of  $d + 1$  Poisson solvers per time steps, and does not exhibit pronounced boundary layer effects. We prove error estimates for both velocity and pressure under realistic regularity conditions via a variational approach, and illustrate the performance with several numerical experiments. Additionally, we apply the gauge-Uzawa into more complicate problems which are Boussineq equations [4] and the non-steady density Navier-Stokes equations[6].

## GAUGE-UZAWA FINITE ELEMENT METHOD(GU-FEM)

GU-FEM [3,5] for the Navier-Stokes equations can be written as following:

Start with  $s_h^0 = 0$  and  $\mathbf{u}_h^0$  as a solution of  $\langle \mathbf{u}_h^0, \mathbf{w}_h \rangle = \langle \mathbf{u}^0, \mathbf{w}_h \rangle$  for all  $\mathbf{w}_h \in \mathbb{V}_h$ .

**Step 1:** Find  $\hat{\mathbf{u}}_h^{n+1} \in \mathbb{V}_h$  as the solution of

$$\begin{aligned} \tau^{-1} \langle \hat{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n, \mathbf{w}_h \rangle + \mathfrak{N}_h(\mathbf{u}_h^n, \hat{\mathbf{u}}_h^{n+1}, \mathbf{w}_h) + \mu \langle \nabla \hat{\mathbf{u}}_h^{n+1}, \nabla \mathbf{w}_h \rangle \\ - \mu \langle s_h^n, \operatorname{div} \mathbf{w}_h \rangle = \langle \mathbf{f}(t^{n+1}), \mathbf{w}_h \rangle, \quad \forall \mathbf{w}_h \in \mathbb{V}_h. \end{aligned} \quad (1)$$

**Step 2:** Find  $\rho_h^{n+1} \in \mathbb{P}_h$  as the solution of

$$\langle \nabla \rho_h^{n+1}, \nabla \psi_h \rangle = \langle \operatorname{div} \hat{\mathbf{u}}_h^{n+1}, \psi_h \rangle, \quad \forall \psi_h \in \mathbb{P}_h. \quad (2)$$

**Step 3:** Update  $s_h^{n+1} \in \mathbb{P}_h$  according to

$$\langle s_h^{n+1}, q_h \rangle = \langle s_h^n, q_h \rangle - \langle \operatorname{div} \hat{\mathbf{u}}_h^{n+1}, q_h \rangle, \quad \forall q_h \in \mathbb{P}_h. \quad (3)$$

**Step 4:** Update  $\mathbf{u}_h^{n+1} \in \mathbb{V}_h + \nabla \mathbb{P}_h$  according to

$$\mathbf{u}_h^{n+1} = \widehat{\mathbf{u}}_h^{n+1} + \nabla \rho_h^{n+1}. \quad (4)$$

Whenever necessary, one may seek pressure as compute

$$p_h^{n+1} = \mu s_h^{n+1} - \tau^{-1} \rho_h^{n+1}. \quad (5)$$

## THE MAIN RESULTS

We now summarize theoretical results of GU-FEM [3]. The GU-FEM is **unconditionally stable** in the sense that, for all  $\tau > 0$ , the following a priori bound holds:

$$\begin{aligned} \|\mathbf{u}_h^{N+1}\|_0^2 + \sum_{n=0}^N \|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\|_0^2 + \frac{\mu\tau}{2} \sum_{n=0}^N \|\nabla \widehat{\mathbf{u}}_h^{n+1}\|_0^2 \\ + 2 \sum_{n=0}^N \|\nabla \rho_h^{n+1}\|_0^2 + \mu\tau \|s_h^{N+1}\|_0^2 \leq \|\mathbf{u}_h^0\|_0^2 + C\tau \sum_{n=0}^N \|\mathbf{f}(t^{n+1})\|_{-1}^2, \end{aligned} \quad (6)$$

and the GU-FEM hold following error bound with realistic regularity assumptions:

$$\begin{aligned} \tau \sum_{n=0}^N \left( \|\nabla (\mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}_h^{n+1})\|_0^2 + \|p(t^{n+1}) - p_h^{n+1}\|_0^2 \right) \leq C(\tau + h^2), \\ \tau \sum_{n=0}^N \left( \|\mathbf{u}(t^{n+1}) - \mathbf{u}_h^{n+1}\|_0^2 + \|\mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}_h^{n+1}\|_0^2 \right) \leq C(\tau + h^2)^2. \end{aligned}$$

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