# Blow-up of the viscous heat-conducting compressible flow

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**Abstract**

In this paper, we consider the following equations for a compressible fluid in $\mathbb{R}^n \times \mathbb{R}_+$ $(n \geq 1)$:

$$\rho_t + \text{ div } (\rho u) = 0, \tag{1}$$

$$(\rho u)_t + \text{div}(\rho u \otimes u) + Lu + \nabla p = 0, \tag{2}$$

$$(\rho e)_t + \text{div}(\rho e u) - \kappa \Delta \theta + p\, \text{div} u = Q(\nabla u), \tag{3}$$

where

$$Lu = -\mu \Delta u - (\lambda + \mu) \nabla \text{div}\, u, \quad \text{and} \quad Q(\nabla u) = \frac{\mu}{2} |\nabla u + \nabla^t u|^2 + \lambda (\text{div}\, u)^2.$$

Here $\rho = \rho(x,t)$, $u = (u_1, \cdots, u_n)$, $\theta$, $p$ and $e$ denote the density, velocity, absolute temperature, pressure and specific internal energy per unit mass, respectively. $\nabla^t u$ is the transpose of $\nabla u$. If we denote the total energy per unit mass $E$ by $E = \frac{1}{2}|u|^2 + e$, then the energy equation (3) can be rewritten by

$$(\rho E)_t + \text{div}\, (\rho E + up) = -Lu \cdot u + Q(\nabla u). \tag{4}$$

The viscosity coefficient $\mu$ and $\lambda$ are assumed to be constant satisfying $\mu \geq 0, \quad \lambda + \frac{2}{n}\mu \geq 0$ from the physical point of view. We also denote by $\kappa \geq 0$ the coefficient of heat conduction.

If $\mu = \lambda = \kappa = 0$, then we call the equations as compressible Euler equations for gas, On the other hand, if $\mu > 0$ and $\lambda + \frac{2}{n}\mu \geq 0$, then we call the equations as compressible Navier-Stokes equations. In particular, we call the equations as heat-conducting compressible Navier-Stokes equations if $\mu > 0$, $\lambda + \frac{2}{n}\mu \geq 0$ and $\kappa > 0$. A polytropic gas is a gas satisfying the following state of equations:

$$p/\rho = R\theta, \quad e = c_\nu \theta \quad \text{and} \quad p/\rho = A \exp(S/c_\nu)\, \rho^{\gamma - 1}, \tag{5}$$

where $R > 0$ is the gas constant, $A$ a positive constant of absolute value, $\gamma > 1$ the ratio of specific heats, $c_\nu = \frac{R}{\gamma - 1}$ the specific heat at constant volume and $S$ the entropy.

The blow-up of smooth solutions of compressible Euler equations has been studied by several mathematicians. In 1985[4], T. C. Sideris showed that the life span $T$ of the $C^1$ solution of the compressible Euler equations is finite when the initial data is constant outside a bounded set and the initial flow velocity is sufficiently large (super-sonic) in some region. In 1998[5], Z. Xin showed, in a different way from [4], the blow-up result for the compressible Euler equations, when the initial density and initial velocity have compact supports. In the paper, he also showed the blow-up of smooth solution for the compressible Navier-Stokes equations for polytropic gas with zero heat conduction (that is, $\kappa = 0$), when the initial density has compact support. His theorem was derived independently of the size of data, but his point of view cannot be applied for $\kappa > 0$, since in his argument the estimation for the lower bound of entropy or time decay of total pressure is strongly necessary, which seems hard to be obtained for the case $\kappa > 0$.

As for the positive result, one may refer to [1]. In the paper [1], the authors showed the local existence of the unique strong solutions of the compressible Navier-Stokes equations (1)-(3) with $n = 3,\ \kappa \geq 0$ and nonnegative density. In particular, they obtained for $\kappa > 0$ that there exists a finite time $T_* > 0$ such that for some $3 < q \leq 6$

$$\begin{aligned} &\rho \in C([0,T_*]; H^1 \cap W^{1,q}), \quad \rho_t \in C([0,T_*]; L^2 \cap L^q), \\ &(u,e) \in C([0,T_*]; H_0^1 \cap H^2), \quad (u_t, e_t) \in L^2(0,T_*; H_0^1). \end{aligned} \tag{6}$$

In this paper, we extend the Xin's blow-up result to the heat-conducting compressible Navier-Stokes equations, that is, for the case $\kappa > 0$ in view of the regularity (6) and hence we provide a sufficient condition for the local result of [1].

Before stating our main theorem, we introduce some notations. We denote by $B_R = B_R(0)$ the ball in $\mathbb{R}^n$ of radius $R$ centered at the origin. We will use several physical quantities:

$$\begin{aligned} m(t) &= \int_{n} \rho(x,t)dx \quad \text{(total mass)}, \\ M(t) &= \int_{n} \rho(x,t)|x|^2 dx \quad \text{(second moment)}, \\ A(t) &= \int_{n} \rho(x,t)u(x,t)\cdot x dx \quad \text{(radial component of momentum)}, \\ \mathcal{E}(t) &= \int_{n} \rho(x,t)E(x,t)\, dx \quad \text{(total energy)} \\ P(t) &= \int_{n} p(x,t)\, dx \quad \text{(total pressure)}. \end{aligned}$$

We always assume that $m(0), M(0), |A(0)|, \mathcal{E}(0) < \infty$ and $m(0) > 0,\ \mathcal{E}(0) > 0$.

For the proof of blow-up, we have only to prove the following theorem.

**Theorem 1** *We assume* $\mu > 0$, $\lambda + \frac{2}{n}\mu > 0$, $1 \leq n \leq 3$ *and* $\kappa \geq 0$. *Let* $\gamma > 1$ *and* $T > 0$. *Suppose that for some* $q > \max(2,n)$

$$\begin{aligned} &\rho \in C([0,T]; H^1 \cap W^{1,q}), \quad \rho_t \in C([0,T]; L^2 \cap L^q), \\ &(u,e) \in C([0,T]; H_0^1 \cap H^2), \quad (u_t, e_t) \in L^2. \end{aligned} \tag{7}$$

*is a solution to the cauchy problem* (1), (2) *and* (3) *with initial data* $(\rho_0, u_0, e_0)$. *Suppose that the initial density* $\rho_0$ *is compactly supported in a ball* $B_{R_0}$. *Then we have*

$$R_0^2 \geq \frac{M(0)}{m(0)} + 2\frac{A(0)}{m(0)}T + \min(2, n(\gamma-1))\frac{\mathcal{E}(0)}{m(0)}T^2. \tag{8}$$

The restriction of dimension can be removed by an appropriate choice of Sobolev spaces guaranteeing the continuity of the solution. For example, we can take $C^1([0,T]; H^k)$ for $k > 2 + [\frac{n}{2}]$ as in [5].

Let $T^*$ be the life span of the solution $(\rho, u, e)$. Then since $m(0)$ and $\mathcal{E}(0)$ are strictly positive, the theorem above implies that $T^*$ should be finite for $\gamma > 1$. It also shows the exact relationship between the size of support and the life span. For example, the range of life span can be extended as the initial support of density become larger. Hence, from the relation, one can expect the global existence of smooth solution of compressible Navier-Stokes equations in case that the initial density is positive but has a decay at infinity in the sense of $M(0) < \infty$.

However even in this case, we show that there is no global solution with $u$ having a little bit fast decay as time goes on as follows:

**Theorem 2** *Suppose that* $(\rho, u, e)$ *is the solution of* (1), (2),(3) *satisfying* (7) *and initial density* $\rho_0$ *is not compactly supported and has* $M(0) < \infty$. *Then there is no global solution of regularity* (7) *with* $T = \infty$ *such that*

$$\limsup_{t\to\infty} \left\| \frac{t}{1+|x|^2} u(x,t) \cdot x \right\|_{L^\infty} < 1. \tag{9}$$

In view of the parabolic scaling $\alpha u(\alpha x, \alpha^2 t)$, it is expected for the global solution with the density away from zero that

$$\limsup_{t\to\infty} \left\| \frac{(1+t)}{1+|x|} u(x,t) \right\|_{L^\infty} \le c,$$

where the constant $c$ can be chosen to be strictly smaller than 1 under a smallness assumption for initial data, rewriting the equation (1)-(3) with $(\tilde{u}, \tilde{e}) = (\frac{1+t}{1+|x|}u, \frac{1+t}{1+|x|}e)$ and using the usual energy estimate for $(\tilde{u}, \tilde{e})$ and elliptic regularity as in [3]. However Theorem 2 shows that even though the bound (9) seems to be reasonable for a density away from zero, the global existence satisfying (9) is impossible for the initial density having a decay at infinity in the sense of $M(0) < \infty$, no matter how small the data is.

Our proof is based on more elementary argument like integration by parts, energy estimate and Gronwall's inequality than in [5]. The key idea is to control the lower bound of the second moment of solution by the evolution of total energy $\mathcal{E}$ via the total radial component of momentum $A$. The control of second moment by total energy enables us not to rely on the lower bound of entropy nor on the time decay of the total pressure $P$. The argument can easily give another proof for the compressible Euler equations and also for the Korteweg type compressible fluid of non-isothermal case if the initial density is compactly supported (see [2] for the later). We leave the details of proof for the later two cases to the readers.

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