

# A collocation least-squares approximation for second-order elliptic partial differential equations

Sangdong Kim<sup>1</sup>, Jungho Yoon<sup>2</sup>, and Changho Kim<sup>3</sup>

<sup>1</sup> Department of Mathematics, Kyungpook National Univ.,  
Taegu 702-701, Korea  
(skim@knu.ac.kr)

<sup>2</sup> Department of Mathematics, Ewha Women's Univ.,  
Seoul, Korea  
(yoon@ewha.ac.kr)

<sup>3</sup> Department of Mathematics and Computer Science, Konkuk Univ.,  
Chungju, Chungbuk, 380-701, Korea  
(kimchang@kku.ac.kr)

## Abstract

Let us consider a second-order elliptic boundary value problem defined on the square  $\Omega := (-1, 1)^2$ :

$$\begin{cases} -\nabla \cdot \mathbf{A} \nabla p + \mathbf{b} \cdot \nabla \mathbf{p} + c_0 p = f, & \text{in } \Omega, \\ p = 0, & \text{on } \Gamma_D, \\ \mathbf{n} \cdot \mathbf{A} \nabla \mathbf{p} = 0, & \text{on } \Gamma_N \end{cases} \quad (1)$$

where  $\partial\Omega = \Gamma_D \cup \Gamma_N$  denotes the boundary of  $\Omega$ ,  $f$  is a given continuous function,  $\mathbf{b}$  and  $c_0$  are given constant vector and nonzero scalar respectively, and  $\mathbf{n}$  is the outward unit vector normal to the boundary. Further, we assume that  $\mathbf{A}$  is a constant symmetric positive definite matrix. For simplicity, we assume that the Dirichlet part of boundary consists of

$$\Gamma_D := (\{-1\} \times [-1, 1]) \cup ([-1, 1] \times \{-1\}). \quad (2)$$

One category of the numerical methods to solve (1) is to use least-squares methodology combined with finite element methods (see further references, for example, [1], [5], [6], [7], etc). In particular, the finite element FOSLS methodology developed in [1] has been known as successful for a use of iterative methods like multigrid method or conjugate gradient method. An alternative approach is to combine FOSLS scheme with very accurate pseudospectral methods, which is reported in [9], using polynomials for approximating the velocity and velocity flux at Legendre-Gaus-Lobatto(=: LGL) or Chebyshev-Gauss-Lobatto(=: CGL) points (see [2], [3], [10], etc). The motivation of this talk is to discuss the way of convergence theoretically when the boundary value problem (1) is solved by a collocation method using any collocation points and to show similar convergence phenomena as pseudospectral least-squares methods possess. Therefore we need to investigate the spectral convergence phenomena measured by Sobolev norm of the problem collocating FOSLS scheme corresponding to (1) at arbitrary chosen scattered points. For this goal, we need to establish a new quadrature formula: For a given set of scattered points  $\{\mathbf{s}_j\}_{j=1}^{M+1}$  in  $\Omega$ , we establish a quadrature formula on  $\Omega$ , which is exact for polynomials of degree less than or equal to  $K$  by using  $(M+1)$  scattered points with  $(M+1) \geq \dim \mathcal{P}_K(\Omega)$  where  $\mathcal{P}_K(\Omega)$  is the polynomial space. The key point in this numerical quadrature is to find a unique admissible vector  $\left(a(\mathbf{t}, \mathbf{s}_l)\right)_{l=0}^M$  such that

$$\sum_{l=0}^M p(\mathbf{s}_l) a(\mathbf{t}, \mathbf{s}_l) = p(\mathbf{t}), \quad \forall p \in \mathcal{P}_K(\Omega) \quad (3)$$

so that

$$\int_{\Omega} p(\mathbf{t})w(\mathbf{t}) dt = \sum_{j=0}^M p(\mathbf{s}_j)\hat{w}_j \quad \forall p \in \mathcal{P}_K(\Omega) \quad (4)$$

for some suitable weights  $\hat{w}_j$ ,  $j = 0, \dots, M$ . In fact, the system (4) is undetermined. However, it can be uniquely determined by making the kernel  $a(t, \mathbf{s}_l)$  in (3) be defined locally. This quadrature formula will be turned out same as, so called, Clenshaw-Curtis quadrature rule when it is restricted to the interval  $[-1, 1]$  and CGL nodes are chosen. A similar one dimensional quadrature formula was developed in [8] in order to show the convergence occurred in a collocation method applied to a second-order elliptic partial differential equation.

The other aim of this talk is to compare the reported numeric results in [9] with the numeric results from collocation least-squares methods applied to a first-order system corresponding to (1) using collocation grid different from the one used to represent the solution. This approach may be classified as a super consistent collocation method which is also very accurate to approximate various kind of differential equations(see [4]). For this goal, we will use two sets of nodes: *representation grid* and the *collocation grid*. Hence, it can be possible to consider using LGL(or CGL) representation grid with CGL (or LGL) collocation points and to measure the convergence in Legendre weighted Sobolev norms (not in Chebyshev weighted Sobolev norms) employing the developed numerical quadrature.

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