decay bounds for an anisotropic penetrative Darcy flow

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Abstract

This paper considers the time-dependent anisotropic penetrative convective Darcy flow in a semi-infinite cylinder. An exponential decay estimate in terms of the distance from the finite end of the cylinder is obtained from a differential inequality for an energy integral. The paper also indicates how to bound the total energy.

1 Introduction

We consider the time-dependent anisotropic penetrative convective Darcy flow in a semi-infinite cylinder. The flow is discussed by Straughan & Walker [1]. Under time-dependent data at the finite end of the cylinder and appropriate homogeneous initial–boundary conditions on the lateral surface of the cylinder prescribed, we investigate Saint-Venant type decay of solutions as the distance from the finite end of the cylinder tends to infinity. Decay results in porous medium have been studied by Payne & Song [2], Song [3]. For a survey of Saint-Venant type spatial decay results, see Horgan & Knowles [4], Horgan [5, 6].

Let R denote the interior of a semi-infinite cylindrical pipe of arbitrary cross section and ∂R its boundary. We assume the generators parallel to the x_3 axis and that the entry section of the cylinder lies in the plane $x_3 = 0$. Denoting the cross section of the pipe by D, the closure of D by $\overline{D}$, and its boundary by ∂D, we introduce the notation:

$$R_z = \{(x_1, x_2, x_3)|(x_1, x_2) \in D,\ x_3 > z \geq 0\},$$

$$D_z = \{(x_1, x_2, x_3)|(x_1, x_2) \in D,\ x_3 = z\}.$$

Clearly, $R_0 \equiv R$ and $D_0 \equiv D$.

The velocity field $u_i(\mathbf{x}, t)\,(i = 1, 2, 3)$, the pressure $p(\mathbf{x}, t)$, and the temperature $T(\mathbf{x}, t)$ for the anisotropic penetrative convective Darcy flow in the pipe are assumed to be classical solutions of

$$\begin{aligned}
M_{ij}u_j &= -p_{,i} + f_i(\mathbf{x})T + g_i(\mathbf{x})T^2 \quad \text{in } R \times \{t > 0\}, && (1.1)\\
u_{j,j} &= 0 \quad \text{in } R \times \{t > 0\}, && (1.2)\\
\frac{\partial T}{\partial t} + u_i T_{,i} &= \Delta T \quad \text{in } R \times \{t > 0\}, && (1.3)\\
u_\alpha n_\alpha &= 0 \quad \text{on} \quad \partial D \times \{x_3 > 0\} \times \{t \geq 0\}, && (1.4)\\
u_3(x_1, x_2, 0, t) &= q(x_1, x_2, t) \quad \text{in } \overline{D} \times \{t > 0\}, && (1.5)\\
T(x_1, x_2, x_3, t) &= 0 \quad \text{on} \quad \partial D \times \{x_3 > 0\} \times \{t \geq 0\}, && (1.6)\\
T(x_1, x_2, 0, t) &= h(x_1, x_2, t) \quad \text{in} \quad \overline{D} \times \{t > 0\}, && (1.7)\\
T(x_1, x_2, x_3, 0) &= 0 \quad \text{in } R \times \{t = 0\}, && (1.8)
\end{aligned}$$

where n_α $(\alpha = 1, 2)$ is the component of the unit normal vector on $\partial D \times \{x_3 > 0\}$, f_i and g_i are given vector functions, and a comma is used to denote partial differentiation. The summation convention of summing in any term over a repeated spatial index, Latin subscripts ranging from 1 to 3 and Greek subscript (α) from 1 to 2 is adopted.

In view of (1.2) and (1.4), the area mean value of u_3 is the same over each cross section. If u_3 is to vanish as $x_3 \to \infty$, it is necessary then that for all t

$$\int_{D_z} q dA = 0. \tag{1.9}$$

We assume that the tensor coefficients M_{ij} are such that for arbitrary vectors ξ_i

$$0 < m\xi_i\xi_i \leq M_{ij}\xi_i\xi_j, \quad \sqrt{M_{ki}M_{kj}\xi_i\xi_j} \leq M\sqrt{\xi_k\xi_k}. \tag{1.10}$$

2 Decay bounds

In order to establish an decay bound for the solutions of (1.1)–(1.8), we consider an energy integral

$$F(z, t) = k\int_0^t\int_{R_z} T_{,i}T_{,i}\, dxd\eta + \int_0^t\int_{R_z} M_{ij}u_ju_i\, dxd\eta, \tag{2.1}$$

where k is a positive constant to be chosen later. Integrating by parts, using (1.1)–(1.8) and dropping a negative term, we have

$$\begin{aligned}
F(z, t) &= -k\int_0^t\int_{D_z} TT_{,3}\, dAd\eta - k\int_0^t\int_{R_z} T(T_{,\eta} + u_iT_{,i})\, dxd\eta\\
&\quad + \int_0^t\int_{R_z} u_i(-p_{,i} + f_iT + g_iT^2)\, dxd\eta\\
&\leq -k\int_0^t\int_{D_z} TT_{,3}\, dAd\eta + \frac{k}{2}\int_0^t\int_{D_z} u_3T^2\, dAd\eta + \int_0^t\int_{R_z} f_iu_iT\, dxd\eta\\
&\quad + \int_0^t\int_{R_z} g_iu_iT^2\, dxd\eta + \int_0^t\int_{D_z} pu_3\, dAd\eta = I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{2.2}$$

On applying the Schwarz inequality, the Poincaré inequality and the arithmetic-geometric mean (AG) inequality to I_n $(n = 1, 2, 3, 4)$, we obtain for arbitrary positive ϵ_1 and ϵ_2

$$\begin{aligned} I_1 &\le k\left(\int_0^t\int_{D_z} T^2\,dAd\eta\right)^{1/2}\left(\int_0^t\int_{D_z} T_{,3}^2\,dAd\eta\right)^{1/2} \\ &\le k\left(\frac{1}{\lambda}\int_0^t\int_{D_z} T_{,\alpha}T_{,\alpha}\,dAd\eta\right)^{1/2}\left(\int_0^t\int_{D_z} T_{,3}^2\,dAd\eta\right)^{1/2} \qquad (2.3)\\ &\le \frac{k}{2\sqrt{\lambda}}\left(\int_0^t\int_{D_z} T_{,\alpha}T_{,\alpha}\,dAd\eta + \int_0^t\int_{D_z} T_{,3}^2\,dAd\eta\right) = \frac{k}{2\sqrt{\lambda}}\int_0^t\int_{D_z} T_{,i}T_{,i}\,dAd\eta, \end{aligned}$$

$$I_2 \le \frac{kT_M}{4\sqrt{\lambda}}\left(\int_0^t\int_{D_z} u_3^2\,dAd\eta + \int_0^t\int_{D_z} T_{,\alpha}T_{,\alpha}\,dAd\eta\right), \quad T_M = \max_{D\times\{t>0\}} h(x_1, x_2, t), \tag{2.4}$$

$$I_3 \le \frac{1}{2\sqrt{\lambda}}\left(\epsilon_1\int_0^t\int_{R_z} u_iu_i\,dxd\eta + \epsilon_1^{-1}f^2\int_0^t\int_{R_z} T_{,i}T_{,i}\,dxd\eta\right), \quad f = \max\sqrt{f_if_i}, \tag{2.5}$$

$$I_4 \le \frac{T_M}{2\sqrt{\lambda}}\left(\epsilon_2\int_0^t\int_{R_z} u_iu_i\,dxd\eta + \epsilon_2^{-1}g^2\int_0^t\int_{R_z} T_{,i}T_{,i}\,dxd\eta\right), \quad g = \max\sqrt{g_ig_i}, \tag{2.6}$$

λ being the smallest eigenvalue of

$$\Delta v + \lambda v = 0 \text{ in } D_z,\ v = 0 \text{ on } \partial D_z. \tag{2.7}$$

In (2.4) and (2.6) we use a maximum principle for T in R, provided u_i is uniformly bounded in R.

Turning to bounding I_5, we introduce a function H defined as the solution of

$$H_{,\alpha\alpha} = u_3 \quad \text{in} \quad D_z\times\{t>0\}, \qquad \frac{\partial H}{\partial n} = 0 \quad \text{on} \quad \partial D_z\times\{t>0\}. \tag{2.8}$$

Such functions exist (defined up to an arbitrary constant) since for all $t > 0$, $\int_{D_z} u_3 dA = 0$. Now by means of this auxiliary function, we have

$$I_5 \le \left(\frac{1}{\mu}\int_0^t\int_{D_z} u_3^2\,dAd\eta\right)^{1/2}\left[\left(M\int_0^t\int_{D_z} u_iu_i\,dAd\eta\right)^{1/2} + \frac{f + gT_M}{\sqrt{\lambda}}\left(\int_0^t\int_{D_z} T_{,\alpha}T_{,\alpha}\,dAd\eta\right)^{1/2}\right], \tag{2.9}$$

where in the last step we have used the inequality [see Payne & Song [7, Eq. (2.18)]

$$\int_0^t\int_{D_z} H_{,\alpha}H_{,\alpha}\,dAd\eta \le \frac{1}{\mu}\int_0^t\int_{D_z} u_3^2\,dAd\eta, \tag{2.10}$$

μ being the first nonzero eigenvalue of

$$\psi_{,\alpha\alpha} + \mu\psi = 0 \text{ in } D_z,\ \partial\psi/\partial n = 0 \text{ on } \partial D_z. \tag{2.11}$$

Substituting the bounds for five integrals I_n into (2.2) and using (1.10), we have

$$\begin{aligned} &\left(k - \frac{f^2\epsilon_1^{-1}}{2\sqrt{\lambda}} - \frac{g^2T_M\epsilon_2^{-1}}{2\sqrt{\lambda}}\right)\int_0^t\int_{R_z} T_{,i}T_{,i}\,dxd\eta + m\left(1 - \frac{\epsilon_1}{2m\sqrt{\lambda}} - \frac{T_M\epsilon_2}{2m\sqrt{\lambda}}\right)\int_0^t\int_{R_z} u_iu_i\,dxd\eta \\ &\quad\le \frac{k}{2\sqrt{\lambda}}\int_0^t\int_{D_z} T_{,i}T_{,i}\,dAd\eta + \frac{kT_M}{4\sqrt{\lambda}}\left(\int_0^t\int_{D_z} T_{,\alpha}T_{,\alpha}\,dAd\eta + \int_0^t\int_{D_z} u_3^2\,dAd\eta\right) \\ &\qquad + \left(\frac{1}{\mu}\int_0^t\int_{D_z} u_3^2\,dAd\eta\right)^{1/2}\left[M\left(\int_0^t\int_{D_z} u_iu_i\,dAd\eta\right)^{1/2} + \frac{f + gT_M}{\sqrt{\lambda}}\left(\int_0^t\int_{D_z} T_{,\alpha}T_{,\alpha}\,dAd\eta\right)^{1/2}\right]. \end{aligned} \tag{2.12}$$

For convenience, we choose

$$\epsilon_1 = \frac{m\sqrt{\lambda}}{2},\ \epsilon_2 = \frac{m\sqrt{\lambda}}{2T_M},\ k = \frac{3f^2}{2m\lambda} + \frac{g^2T_M^2}{m\lambda}. \tag{2.13}$$

Then inserting into (2.12) we may obtain

$$\begin{aligned} E(z,t) &= \frac{f^2}{2m\lambda}\int_0^t\int_{R_z} T_{,i}T_{,i}\,dxd\eta + \frac{m}{2}\int_0^t\int_{R_z} u_iu_i\,dxd\eta \\ &\le -\gamma\frac{\partial E}{\partial z}, \end{aligned} \tag{2.14}$$

where

$$\gamma = \frac{km\sqrt{\lambda}}{f^2} + \frac{kT_Mm\sqrt{\lambda}}{2f^2} + \frac{kT_M}{2m\sqrt{\lambda}} + \frac{2M}{m\sqrt{\mu}} + \frac{2(f+gT_M)}{\sqrt{\mu}f}. \tag{2.15}$$

Integrating (2.14) yields

$$E(z,t) \le E(0,t)e^{-z/\gamma}. \tag{2.16}$$

Theorem 1 *Let (u_i, T) be a solution of* (1.1) *and (u_i, T) tends to $o(1)$ as $z \to \infty$, then for fixed t, we have*

$$\frac{f^2}{2m\lambda}\int_0^t\int_{R_z} T_{,i}T_{,i}\,dxd\eta + \frac{m}{2}\int_0^t\int_{R_z} u_iu_i\,dxd\eta \le E(0,t)e^{-z/\gamma},\ \textit{where}\gamma\ \textit{is given in}(2.15). \tag{2.17}$$

It remains to bound $E(0,t)$ in terms of data.

References

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