GLOBAL COUPLING EFFECTS ON A FREE BOUNDARY PROBLEM FOR THREE-COMPONENT REACTION-DIFFUSION SYSTEM

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ABSTRACT

In this paper, we consider three-component reaction-diffusion system. With an integral condition and a global coupling, this system gives us an interesting free boundary problem. We shall examine the occurrence of a Hopf bifurcation and the stability of solutions as the global coupling constant varies. The main result is that a Hopf bifurcation occurs for weak global coupling.

INTRODUCTION

We now consider three-component reaction-diffusion system that describe the interaction of two inhibitors v and w and one activator u in [3,4]:

$$\begin{cases} \varepsilon\sigma u_t = \varepsilon^2 u_{xx} + f(u, v, < w >), \\ v_t = v_{xx} + g(u, v, < w >), \\ 0 = \int_0^L h(u, v, < w >)dx, \quad x \in (0, L),\, t > 0 \end{cases} \tag{1}$$

with the Neumann boundary conditions $v_x(0,t) = 0 = v_x(L,t)$. The nonlinear terms are taken the piecewise linear case of FitzHugh-Nagumo type[1,2] which are

$$f = -u + H(u - a_0) - v,\ g = \mu u - v - < w >,\ h = \kappa_1 - \kappa_2 v - w \tag{2}$$

where κ_2 is a global coupling constant and μ, a_0 and κ_1 are all positive constants. A spatial average of w is defined by $< w >= \frac{1}{L}\int_0^L w dx$.

A single interface problem with the global coupling is obtained by

$$\begin{cases} v_t + Av = \mu H(x - s) + \kappa_2 q - \kappa_1, & (x \in (0, L) \setminus \{s\},\, t > 0) \\ s'(t) = \frac{1}{\sigma} C(v(s(t), t), q(t)), & (t > 0) \\ q'(t) = -(\mu + 1 - \kappa_2)q + \mu(1 - \frac{s}{L}) - \kappa_1\,, & (t > 0) \\ v(x, 0) = v_0(x)\,,\ s(0) = s_0\,,\ q(0) = q_0 \end{cases} \tag{3}$$

where $A = -\frac{d^2}{dx^2} + (\mu + 1)$ and $q =< v >$.

REGULARIZED EQUATION OF MOTION

Define $g : [0, L] \times [0, L] \times \mathbb{C} \longrightarrow \mathbb{C}$ by

$$g(x, s, q) := A^{-1}(\mu H(x - s) + \kappa_2 q - \kappa_1)$$

and $\gamma : [0, L] \times \mathbb{C} \longrightarrow \mathbb{C}$ by

$$\gamma(s, q) := g(s, s, q)\,.$$

We decompose a solution (v, s, q) of (3) into two parts by defining

$$u(t)(x) := v(x, t) - g(x, s(t), q(t))\,.$$

The initial value problem for (u, s, q) can then be written as

$$\begin{aligned} &\frac{d}{dt}(u, s, q) + \widetilde{A}(u, s, q) = f(u, s, q) \\ &(u, s, q)(0) = (u(0), s(0), q(0)) = (u_0, s_0, q_0) \end{aligned} \tag{4}$$

of a differential equation in a Banach space $\widetilde{X}$ of the form $\widetilde{X} := X \times \mathbb{R} \times \mathbb{C}$. The operator $\widetilde{A}$, represented in matrix form

$$\widetilde{A} = \begin{pmatrix} A & -\frac{\mu\,\kappa_2}{L(\mu+1)} & -\frac{\kappa_2(\mu+1-\kappa_2)}{\mu+1} \\ 0 & 0 & 0 \\ 0 & \frac{\mu}{L} & \mu + 1 - \kappa_2 \end{pmatrix}.$$

The nonlinear forcing term f defined on the set

$$W = \{(u, s, q) \in C^1([0, L]) \times \mathbb{R} \times \mathbb{C}\,:\, u(s) + \gamma(s, q) \in I\} \subset_{\text{open}} C^1([0, L]) \times \mathbb{R} \times \mathbb{C}$$

is as follows :

$$f(u, s\,, q\,) = \begin{pmatrix} \frac{1}{\sigma}\, f_2(u, s, q) \cdot f_1(s) - \frac{\kappa_2\,(\mu-\kappa_1)}{\mu+1} \\ \frac{1}{\sigma}\, f_2(u, s, q) \\ \mu - \kappa_1 \end{pmatrix},$$

where $f_1 : (0, L) \to X,\quad f_1(s)(x) := \mu\, G(x, s)$ and $f_2 : W \to \mathbb{C},\quad f_2(u, s, q) := C\Big(u(s) + \gamma(s, q), s, q\Big)$,

$$C(u(s) + \gamma(s, q), s, q) = \frac{2\Big(u(s) + \gamma(s, q)\Big) + 2a_0 - 1}{\sqrt{\Big(a_0 + u(s) + \gamma(s, q)\Big)\Big(1 - a_0 - u(s) - \gamma(s, q)\,\Big)}}\,.$$

EXISTENCE OF STEADY-STATES AND LINEARIZED EQUATION OF MOTION

Theorem 1 *(1) Case without global coupling ($\kappa_2 = 0$). Assume that $\frac{1}{2} - a_0 < \frac{\mu-\kappa_1}{\mu+1}$. For all $\sigma > 0$, the problem of (4) has the solution $(0, s^*, q^*)$ with $q^* = \frac{\mu\left(1-\frac{s^*}{L}\right)-\kappa_1}{\mu+1}$ and $s^* \in (0, L)$.*

(2) Case with global coupling ($\kappa_2 > 0$).

(a) Suppose $\kappa_2 = \mu+1$. Then for all $\sigma > 0$ the stationary problem of (4) has the solution $(0, s^*, q^*)$ with $q^* = \frac{1}{2} - a_0 + \frac{\kappa_1}{\mu+1} - \frac{\cosh L\sqrt{\mu+1}(1-\frac{\kappa_1}{\mu})\,\sinh(L\sqrt{\mu+1}\frac{\kappa_1}{\mu})}{(\mu+1)\sinh(L\sqrt{\mu+1})}$ and $s^* = L(1-\frac{\kappa_1}{\mu})$.

(b) Suppose that $\kappa_2 < \mu+1$ and $\frac{1}{2} - a_0 < \frac{\mu-\kappa_1}{\mu+1-\kappa_2}$. For all $\sigma > 0$, the problem of (4) has the stationary solution $(0, s^*, q^*)$ with $q^* = \frac{\mu\left(1-\frac{s^*}{L}\right)-\kappa_1}{\mu+1-\kappa_2}$ and $s^* \in (0, L)$.

(c) For $\mu+1 < \kappa_2 < \infty$, assume that $\frac{1}{2} - a_0 < \frac{\kappa_1}{\kappa_2-\mu-1}$. Then there exists the solution $(0, s^*, q^*)$ such that $q^* = \frac{\mu\left(1-\frac{s^*}{L}\right)-\kappa_1}{\mu+1-\kappa_2}$ and $s^* \in (s_c, L-s_c)$, where

$$s_c = s_c(\kappa_2) = \frac{L}{2} - \frac{1}{2\sqrt{\mu+1}}\ln\left(K+\sqrt{K^2-1}\right),\quad K = \frac{\kappa_2\sinh(L\sqrt{\mu+1})}{L\sqrt{\mu+1}(\kappa_2-\mu-1)}.$$

(3) Case with strong global coupling ($\kappa_2 \uparrow \infty$). Suppose that

$$\tfrac{1}{2} - a_0 < \tfrac{\mu}{2L\,(\mu+1)^{3/2}\,K_\infty}\left(K_\infty\,\ln\left(K_\infty+\sqrt{K_\infty^2-1}\right)-\sqrt{K_\infty^2-1}\right)$$

where $K_\infty = \frac{\sinh(L\sqrt{\mu+1})}{L\sqrt{\mu+1}}$. Then for all $\sigma > 0$, the problem of (4) has the stationary solution $(0, s^*, q^*)$ with $q^* = 0$ and $s^* \in (s_\infty, L-s_\infty)$ where $s_\infty = \lim_{\kappa_2\to\infty} s_c(\kappa_2)$.

For all cases, the linearization of f at $(0, s^*, q^*)$ is

$$Df(0,s^*,q^*)(\hat{u},\hat{s},\hat{q}) = \begin{pmatrix} \frac{4}{\sigma}\left(\hat{u}(s^*)+\gamma_s(s^*,q^*)\hat{s}+\gamma_q(s^*,q^*)\hat{q}\right)\mu G(\cdot,s^*) \\ \frac{4}{\sigma}\left(\hat{u}(s^*)+\gamma_s(s^*,q^*)\hat{s}+\gamma_q(s^*,q^*)\hat{q}\right) \\ 0 \end{pmatrix}.$$

The pair $(0, s^*, q^*)$ corresponds to a unique steady state (v^*, s^*, q^*) of (3) for $\sigma \neq 0$ with $v^*(x) = g(x, s^*, q^*)$.

EFFECTS OF GLOBAL COUPLING FOR A HOPF BIFURCATION

A Hopf bifurcation without global coupling($\kappa_2 = 0$):

We state our main theorem:

Theorem 2 *Suppose that $\frac{1}{2} - a_0 < \frac{\mu-\kappa_1}{\mu+1}$. The problem (4), respectively (3), has stationary solutions (u^*, s^*, q^*) where $u^* = 0$ and $q^* = \frac{\mu\left(1-\frac{s^*}{L}\right)-\kappa_1}{\mu+1}$ respectively (v^*, s^*, q^*) for all $\tau >$*

0. *Then there exists a unique τ^* such that the linearization $-\widetilde{A} + \tau^* B$ has a purely imaginary pair of eigenvalues $\beta > 0$. The point $(0, s^*, q^*, \tau^*)$ is then a Hopf point for (4) and there exists a C^0-curve of nontrivial periodic orbits for (4), respectively (3), bifurcating from $(0, s^*, q^*, \tau^*)$, respectively (v^*, s^*, q^*, τ^*).*

A Hopf bifurcation with global coupling ($\kappa_2 > 0$):
Theorem 3 *Suppose that (i) $\kappa_2 = \mu + 1$ then the problem (4), respectively (3), has a unique stationary solution (u^*, s^*, q^*) where $u^* = 0,\ q^* = \frac{1}{2} - a_0 + \frac{\kappa_1}{\mu+1} - \frac{\cosh\ L\sqrt{\mu+1}(1-\frac{\kappa_1}{\mu})\ \sinh(L\sqrt{\mu+1}\frac{\kappa_1}{\mu})}{(\mu+1)\sinh(L\sqrt{\mu+1})}$ and $s^* = L(1 - \frac{\kappa_1}{\mu})$ respectively (v^*, s^*, q^*) for all $\tau > 0$. (ii) If $0 \leq \kappa_2 < \mu + 1$ and $\frac{1}{2} - a_0 < \frac{\mu-\kappa_1}{\mu+1-\kappa_2}$, the problem (4), respectively (3), has stationary solutions (u^*, s^*, q^*) where $u^* = 0$ and $q^* = \frac{\mu\left(1-\frac{s^*}{L}\right) - \kappa_1}{\mu+1-\kappa_2}$ respectively (v^*, s^*, q^*) for all $\tau > 0$. Then there exists a unique τ^* such that the linearization $-\widetilde{A} + \tau^* B$ has a purely imaginary pair of eigenvalues $\beta > 0$. The point $(0, s^*, q^*, \tau^*)$ is then a Hopf point for (4) and there exists a C^0-curve of nontrivial periodic orbits for (4), respectively (3), bifurcating from $(0, s^*, q^*, \tau^*)$, respectively (v^*, s^*, q^*, τ^*).*

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