Existence of multiple positive radial solutions for p-Laplacian problems on an exterior domain via bifurcation methods

Chan-Gyun Kim , Yong-Hoon Lee and Inbo Sim

Department of Mathematics, Pusan National University, Pusan 609-735, KOREA

Corresponding Author : Yong-Hoon Lee, e-mail yhlee@pusan.ac.kr

ABSTRACT

We find the second positive radial solution for the following p-Laplacian problem:

$$\begin{cases} \text{div}\,(|\nabla u|^{p-2}\nabla u) + K(|x|)u^q = 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \\ u(x) \to \mu > 0, & \text{as } |x| \to \infty, \end{cases}$$

where $\Omega = \{x \in \mathbb{R}^N : |x| > r_0\},\ r_0 > 0,\ N > p > 1,\ q > p-1.$

INTRODUCTION

In this paper, we study the existence, nonexistence and multiplicity of positive radial solutions for the following p-Laplacian problem

$$\text{div}(|\nabla u|^{p-2}\nabla u) + K(|x|)f(u) = 0 \text{ in } \Omega, \tag{P}$$

$$u|_{\partial\Omega} = 0 \text{ and } u \to \mu > 0 \text{ as } |x| \to \infty, \tag{D_1}$$

where $\Omega = \{x \in \mathbb{R}^N : |x| > r_0\},\ r_0 > 0,\ N > p > 1,\ \mu$ a positive real parameter, $K \in C(\Omega, (0,\infty))$ and $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\mathbb{R}_+ = [0,\infty)$.

The present work is motivated by Deng and Li [1]. They considered a semilinear problem of the form

$$\begin{cases} \Delta u + K(x)u^\alpha = 0 \text{ in } \Omega, \\ u > 0 \text{ in},\ u \in H^1_{\text{loc}}(\Omega) \cap C(\bar{\Omega}), \\ u|_{\partial\Omega} = 0,\ u \to \mu > 0 \text{ as } |x| \to \infty, \end{cases} \tag{DL}$$

where $\Omega = \mathbb{R}^N \backslash \omega$ is an exterior domain in $\mathbb{R}^N,\ \omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and $N > 2,\ \alpha > 1$. Among other results, they prove under assumption;

$(K1)$ $K \in C^\alpha_{\text{loc}}(\Omega),\ K \geq 0, K \not\equiv 0$ and there exist $C,\ \epsilon,\ M > 0$ such that $|K(x)| \leq C|x|^{-l}$ for $|x| \geq M$ with $l \geq 2 + \epsilon$

that there exists $\mu^* > 0$ such that (DL) has at least one solution for $\mu \in (0, \mu^*)$ and no solution for $\mu \in (\mu^*, \infty)$. Furthermore, if $K \in L^1(\Omega)$, then the solution at $\mu = \mu^*$ is unique.

We expect the problem may have bifurcation phenomenon of solutions with respect to μ, i.e., there should be one more solution for $\mu \in (0, \mu^*)$. Our first result comes up as follows. Assume

(K) there exists $\gamma > p-1$ such that $\int_{r_0}^\infty r^\gamma K(r)dr < \infty$,

(F_1) f is homogeneous of degree ρ with $\rho > p-1$,

(F_2) $f(u) > 0$ for all $u > 0$,

(F_3) $f_\infty \triangleq \lim_{u\to\infty} \frac{f(u)}{u^{p-1}} = \infty.$

Then, there exist $\mu_0 \geq \mu^* > 0$ such that $(P) + (D_1)$ has at least two positive radial solutions for $\mu \in (0, \mu^*)$, at least one positive radial solution for $\mu \in [\mu^*, \mu_0]$ and no radial solution for $\mu \in (\mu_0, \infty)$.

We notice that a model example of f satisfying conditions $(F_1) - (F_3)$ is $f(u) = u^q$, $q > p-1$. We also notice that this result is partial, since the existence interval $[\mu^*, \mu_0]$ of multiple solutions is not obvious. This is mainly caused by coarse topological structure of solution space. For more details, applying consecutive changes of variables, $r = |x|$, $u(r) = u(|x|)$ and $t = \left(\frac{r}{r_0}\right)^{\frac{-N+p}{p-1}}$, $z(t) = u(r)$, problem $(P) + (D_1)$ is equivalently written as

$$\begin{cases} \varphi_p(z')' + \lambda h(t) f(z) = 0,\ t \in (0,1), \\ z(0) = \mu > 0, z(1) = 0, \end{cases} \tag{1}$$

where $\varphi_p(s) = |s|^{p-2}s$ and h is given by

$$h(t) = \left(\frac{p-1}{N-p}\right)^p r_0^p t^{\frac{-p(N-1)}{N-p}} K\left(r_0 t^{\frac{-(p-1)}{N-p}}\right).$$

We notice that h is singular at $t = 0$ and by condition (K), h satisfies

$$\int_0^1 s^\beta h(s) ds < \infty, \tag{H_1}$$

for some $\beta < p-1$. For more general consideration, we assume that the coefficient function h may be singular at $t = 0$ and/or 1 which satisfies

$$\int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} h(\tau)d\tau\right) ds + \int_{\frac{1}{2}}^1 \varphi_p^{-1}\left(\int_{\frac{1}{2}}^s h(\tau)d\tau\right) ds < \infty. \tag{H}$$

Obviously, we see that condition (H_1) implies condition (H). By using the homogeneity of f and introducing $u(t) = \frac{z(t)}{\mu}$, we can rewrite problem (1) as

$$\begin{cases} \varphi_p(u')' + \lambda h(t) f(u) = 0,\ t \in (0,1), \\ u(0) = a, u(1) = 0, \end{cases} \tag{2}$$

where $a > 0$ fixed, $\lambda = \mu^{\rho-p+1}$ and h satisfies condition (H). Problems (1) and (2) share the same bifurcation phenomena with respect to μ and λ. Since h may not be in $L^1(0,1)$, solutions are not guaranteed of $C^1[0,1]$ so that analysis for the first result has to be performed on $C[0,1]$ space. Lack of controllability of derivatives at the boundary makes difficulty on the computation of fixed point index which causes such partial result for multiplicity. To overcome this difficulty, we introduce weighted space involving with derivatives somehow and do similar analysis for computation of fixed point index. For simplicity, we confine our problem as follows,

$$\text{div}(|\nabla u|^{p-2}\nabla u) + |x|^{-l} f(u) = 0 \text{ in } \Omega, \tag{E}$$

$$u|_{\partial\Omega} = 0 \text{ and } u \to \mu > 0 \text{ as } |x| \to \infty. \tag{D_1}$$

Proofs are employed by global bifurcation theorem of Rabinowitz and fixed point index theory.

THE EXISTENCE OF UNBOUNDED CONTINUUM

Let us consider

$$\begin{cases} -\varphi_p(u'(t))' = \lambda h(t) f(u(t)), \ \ t \in (0,1), \\ u(0) = a, \ u(1) = 0, \end{cases} \tag{P_λ}$$

where $f \in C([0,\infty),[0,\infty))$ and

$$h \in \mathcal{A} \triangleq \left\{ q \in C(0,1) : \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} q(\tau)d\tau\right) ds + \int_{\frac{1}{2}}^1 \varphi_p^{-1}\left(\int_{\frac{1}{2}}^s q(\tau)d\tau\right) ds \right\}.$$

Substituting $v(t) = u(t) - a(1-t)$, we have

$$\begin{cases} -\varphi_p(v'(t) - a)' = \lambda h(t) f(v(t) + a(1-t)), \ \ t \in (0,1), \\ v(0) = 0 = v(1), \end{cases} \tag{P_λ'}$$

Denote $K = \{u \in C_0[0,1] : u \text{ is concave } \}$. Then it is easy to see that K is an order cone. Let the operator $H(\lambda, v) : [0,\infty) \times K \to K$ be defined by

$$H(\lambda,v)(t) = \begin{cases} \int_0^t \varphi_p^{-1}\left(\int_s^{A_{\lambda,v}} \lambda h(\tau) f(v(\tau) + a(1-\tau))d\tau - \varphi_p(a)\right) ds + at, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le t \le A_{\lambda,v}, \\ \int_t^1 \varphi_p^{-1}\left(\int_{A_{\lambda,v}}^s \lambda h(\tau) f(u(\tau) a(1-\tau))d\tau + \varphi_p(a)\right) ds - a(1-t), \\ \qquad\qquad\qquad\qquad\qquad\qquad A_{\lambda,v} \le t \le 1, \end{cases}$$

where

$$\begin{aligned} &\int_0^{A_{\lambda,v}} \varphi_p^{-1}\left(\int_s^{A_{\lambda,v}} \lambda h(\tau) f(v(\tau) + a(1-\tau))d\tau - \varphi_p(a)\right) ds + aA_{\lambda,v} \\ &= \int_{A_{\lambda,v}}^1 \varphi_p^{-1}\left(\int_{A_{\lambda,v}}^s \lambda h(\tau) f(v(\tau) + a(1-\tau))d\tau + \varphi_p(a)\right) ds - a(1 - A_{\lambda,v}). \end{aligned} \tag{3}$$

Assume (H) and $(F_1) - (F_3)$, then by the generalized Picone identity and the properties the p-sine function [3], we obtain the followings;

Theorem 0.1 *$H(\lambda, v)$ is completely continuous on $[0,\infty) \times K$.*

Theorem 0.2 *There exists an unbounded continuum $\mathcal{C}$ bifurcating from $(0, a)$ in the closure of the set of positive solutions of (P_λ) in $[0,\infty) \times K$.*

Lemma 0.3 *Let u be a positive solution of (P_λ). Then there exists $\bar{\lambda} > 0$ such that $\lambda \le \bar{\lambda}$.*

Lemma 0.4 *Let $J = [\alpha, \beta]$ be a compact interval in $(0,\infty)$. Then for all $\lambda \in J$, there exists $M_J > 0$ such that all possible positive solutions u of (P_λ) satisfy $||u||_\infty \le M_J$.*

Theorem 0.5 *There exists $0 < \lambda^* \le \lambda_*$ such that (P_λ) has two positive solutions for $0 < \lambda < \lambda^*$, one positive solution for $\lambda^* \le \lambda \le \lambda_*$, and no positive solution for $\lambda > \lambda_*$.*

GLOBAL EXISTENCE RESULT

Let us consider the following p-Laplacian problem,

$$\text{div}(|\nabla u|^{p-2}\nabla u) + |x|^{-l} f(u) = 0 \text{ in } |x| > r_0, \qquad (E)$$

$$u|_{|x|=r_0} = 0 \text{ and } u \to \mu > 0 \text{ as } |x| \to \infty. \qquad (D_1)$$

Condition (K) with $K(|x|) = |x|^{-l}$ corresponds to $l > p$. By transformations $r = |x|,\ u(r) = z(x)$ and $t = \left(\frac{r}{r_0}\right)^{-\frac{(N-p)}{p-1}}$, we obtain

$$\begin{cases} \varphi_p(z')' + \left(\frac{p-1}{N-p}\right)^p r_0^p t^{\frac{-p(N-1)+(p-1)l}{N-p}} f(z) = 0, t \in (0,1) \\ z(0) = \mu > 0,\ z(1) = 0. \end{cases}$$

For $\alpha = \frac{p(N-1)-(p-1)l}{N-p}$, condition $l > p$ yields $\alpha < p$. By transformations $u(t) = \frac{z(t)}{\mu}$ and $v(t) = u(t) - a(1-t)$, we get

$$\begin{cases} \varphi_p(v'(t) - a)' + \lambda t^{-\alpha} f(v(t) + a(1-t)) = 0,\ t \in (0,1), \\ v(0) = 0 = v(1). \end{cases} \qquad (E_\lambda)$$

where $0 < \alpha < p$. Now the aim of our work is to investigate global bifurcation phenomena of positive solutions for problem (E_λ). If $0 < \alpha < 1$, then $h(t) = t^{-\alpha}$ is $L^1(0,1)$. Thus solution space is $C^1[0,1]$ and by typical Leray-Schauder degree argument, in the frame of C^1-topology, the theorem can be proved [2]. In this section, we focus on the case $1 \le \alpha < p$. Define

$$w(t) = \begin{cases} t^{\frac{\alpha-1}{p-1}}, & \text{if } 1 < \alpha < p \\ \min\{(-\ln t)^{\frac{-1}{p-1}}, 1\}, & \text{if } \alpha = 1 \end{cases}$$

and also define weighted solution space $C_w[0,1] = \{u \in C_0[0,1] : wu' \in C[0,1]\}$. Then $(C_w, ||\cdot||_w)$ is a Banach space with $\|u\|_w = \|u\|_\infty + \|wu'\|_\infty$. Let $K = \{u \in C_w | u$ is concave on $(0,1)\}$. By *a priori* estimates of solutions and degree computations on $C_w[0,1]$, we get the second main theorem as follows.

Theorem 0.6 *Assume* $(F_1) - (F_3)$. *Also assume*

(A) $\alpha < p$,

(F_4) f *is nondecreasing.*

Then there exists $\lambda^* > 0$ *such that* (E_λ) *has at least two positive solutions for* $\lambda \in (0, \lambda^*)$, *at least one positive solution for* $\lambda = \lambda^*$ *and no solution for* $\lambda \in (\lambda^*, \infty)$.

REFERENCES

1. Y. Deng and Y. Li, *On the existence of multiple positive solutions for a semilinear problem in exterior domains,* J. of Differential Equations, 181:197–229, 2002.
2. C. Kim and Y. Lee, *Existence of multiple positive solutions for a p-Laplacian problem with L^1 indefinite weight,* Preprint.
3. M. Zhang, *Nonuniform nonresonance of semilinear differential equations,* J. of Differential Equations, 166:33–50, 2000.