

IDENTIFICATION OF POLES AND RESIDUES OF MEROMORPHIC FUNCTIONS WITH FINITE NUMBER OF POLES AND ZEROS AND ITS STABILITY ESTIMATE

Yun-Sung CHUNG¹ and Soon-Yeong CHUNG²

1) *Institute of Mathematical Sciences, Ewha Womans University, Seoul, Korea*

2) *Department of Mathematics, Sogang University, Seoul, KOREA*

Corresponding Author : Yun-Sung CHUNG, ysjung93@ewha.ac.kr

ABSTRACT

In this talk, we propose an algorithm for identifying the poles and the residues of meromorphic functions with finite number of poles and zeros on a simply connected bounded domain from their boundary measurements with some stability results. We prove that the pole locations are shown to be the solutions of some characteristic equation related to the inverse of a Hankel matrix derived from the boundary data and apply this fact to provide a detection algorithm.

INTRODUCTION

A *meromorphic function* is a single-valued function that is analytic in all but possibly a discrete subset of its domain and those exceptional points must be poles and not essential singularities. It is well known that if a meromorphic function $f(z)$ has a finite number of poles and zeros on a simply connected bounded domain D , then it can be represented as the form

$$f(z) = \frac{g(z)}{h(z)} \quad (1)$$

where $g(z)$ and $h(z)$ are polynomials whose zeros are contained in D . Thus the function $f(z)$ is represented as

$$f(z) = \frac{(z - \omega_1)^{k_1} (z - \omega_2)^{k_2} \cdots (z - \omega_M)^{k_M}}{(z - \zeta_1)^{m_1} (z - \zeta_2)^{m_2} \cdots (z - \zeta_N)^{m_N}} \quad (2)$$

$$\begin{aligned} &= \left[\sum_{j=1}^{N_1} \frac{\lambda_{1j}}{z - \alpha_{1j}} \right] + \left[\sum_{j=1}^{N_2} \frac{\lambda_{1j}^{(2)}}{z - \alpha_{2j}} + \frac{\lambda_{2j}^{(2)}}{(z - \alpha_{2j})^2} \right] + \cdots \\ &\quad + \left[\sum_{j=1}^{N_m} \frac{\lambda_{1j}^{(m)}}{z - \alpha_{mj}} + \cdots + \frac{\lambda_{mj}^{(m)}}{(z - \alpha_{mj})^m} \right] + h(z) \quad (3) \end{aligned}$$

for a holomorphic function $h(z)$ and some α_{ij} and $\lambda_{ij}^{(n)} \in D \subset \mathbb{C}$, where α_{ij} , $j = 1, 2, \dots$ is the pole with i th order, $i = 1, 2, \dots$. The main consideration in this talk is to provide an algorithm for identifying all locations α_{ij} and number N_i , $i = 1, 2, \dots, m$ and their corresponding

residues $\lambda_{ij}^{(n)}$ by using the measurements of the value $f(z)$ in ∂D . It is also discussed whether the algorithm is stable under noisy boundary data. To show its stability, we provide a stability estimate under the error measurements.

1 MAIN RESULTS

Let c_n be defined as

$$c_n := \int_{\partial D} z^n f(z) dz, \quad n = 0, 1, 2, \dots \quad (4)$$

and define the $N \times N$ Hankel matrix H_N as

$$H_N := \begin{pmatrix} c_0 & c_1 & \cdots & c_{N-1} \\ c_1 & c_2 & \cdots & c_N \\ \vdots & \vdots & \ddots & \vdots \\ c_{N-1} & c_N & \cdots & c_{2N-2} \end{pmatrix}. \quad (5)$$

The following are the main results of this talk.

Theorem 1

$$\det C_{0,k \times k} \begin{cases} \neq 0, & k = N_1 + 2N_2 + \cdots + mN_m \\ = 0, & k \geq N_1 + 2N_2 + \cdots + mN_m \end{cases} \quad (6)$$

Theorem 2 *Let $K = N_1 + 2N_2 + \cdots + mN_m$. The following are equivalent.*

(1) $l_1, l_2, \dots, l_K \in \mathbb{C}$ satisfy

$$z^K + l_1 z^{K-1} + \cdots + l_{K-1} z + u_K = \prod_{j=1}^{N_1} (z - \alpha_{1j}) \prod_{j=1}^{N_2} (z - \alpha_{2j})^2 \cdots \prod_{j=1}^{N_m} (z - \alpha_{mj})^m. \quad (7)$$

(2) $l_1, l_2, \dots, l_K \in \mathbb{C}$ satisfy

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_{K-1} \\ c_1 & c_2 & \cdots & c_K \\ \vdots & \vdots & \ddots & \vdots \\ c_{K-1} & c_K & \cdots & c_{2K-2} \end{pmatrix} \begin{pmatrix} l_K \\ l_{K-1} \\ \vdots \\ l_1 \end{pmatrix} = \begin{pmatrix} -c_K \\ -c_{K+1} \\ \vdots \\ -c_{2K-1} \end{pmatrix} \quad (8)$$

2 ALGORITHM

Theorem 1 and 2 suggests that if we know an upper bound $K^* \in \mathbb{N}$ for the number $K = N_1 + 2N_2 + \cdots + mN_m$, then one can establish an algorithm to detect the number K and the location α_{ij} of poles as follows:

[I] Using the known data $f(z)$, $z \in \partial D$, compute $c_0, c_1, \cdots, c_{2K^*-1}$.

[II] Starting with $k = K^*$, compute the determinant $\det H_k$ for $k = K^*, K^* - 1, \cdots$ until one first finds an integer K satisfying $\det H_K \neq 0$.

[III] Solve the following equation

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_{K-1} \\ c_1 & c_2 & \cdots & c_K \\ \vdots & \vdots & \ddots & \vdots \\ c_{K-1} & c_K & \cdots & c_{2K-2} \end{pmatrix} \begin{pmatrix} l_K \\ l_{K-1} \\ \vdots \\ l_1 \end{pmatrix} = \begin{pmatrix} -c_K \\ -c_{K+1} \\ \vdots \\ -c_{2K-1} \end{pmatrix} \quad (9)$$

to calculate l_1, l_2, \cdots, l_K .

[IV] Solve the polynomial equation

$$z^K + u_1 z^{K-1} + \cdots + u_K = 0. \quad (10)$$

to obtain M_n roots of order n , say $\zeta_{n,1} \cdots, \zeta_{n,M_1}$, $n = 1, 2, \cdots, m$. Then we have

$$\alpha_{ij} = (\operatorname{Re} \zeta_{i,j}, \operatorname{Im} \zeta_{i,j}) \quad i = 1, 2, \cdots, m \quad (11)$$

where $\operatorname{Re} \zeta$ and $\operatorname{Im} \zeta$ denote the real and the imaginary parts of the complex number ζ , respectively.

The method of determining residues $\lambda_{ij}^{(n)}$ will also be discussed in the talk.

For the practical purpose, it is also important to consider the stability problem - whether the algorithm stable under error measurement, which also will be discussed in the talk.

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